# ON SUMS OF RECIPROCALS OF FIBONACCI AND LUCAS NUMBERS

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In this paper we present some remarkable elementary identities for sums of powers of reciprocals of Fibonacci and Lucas numbers. The Fibonacci numbers are defined for all $n \geq 0$ by the recurrence relation $F_{n+1} = F_n + F_{n-1}$, where $F_0 = 0$ and $F_1 = 1$. The Lucas numbers $L_n$ are defined for all $n \geq 0$ by the same recurrence relation, where $L_0 = 2$ and $L_1 = 1$. The general theorems in this paper include as special cases the following results:

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{F_{4n-2}}, \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^3} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{L_{2n-1}}, \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} = \sqrt{5} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{F_{2n}}, \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{n}{F_{2n}}. \tag{4}$$

Identity (3) appears on page 98 of [1]. Identity (4) is really just the complementary result of (3). Identities (1) and (2) are believed to be new. The above four results are just the first cases of the following theorems.

***Theorem 1:*** For $k = 1, 2, 3, \ldots$, we have

$$\frac{1}{5^k} \sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^{2k+1}} = \frac{1}{(2k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1)\cdots(n+k-1)}{F_{4n-2}}.$$

***Theorem 2:*** For $k = 1, 2, 3, \ldots$, we have

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^{2k+1}} = \frac{1}{(2k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1)\cdots(n+k-1)}{L_{2n-1}}.$$

***Theorem 3:*** For $k = 1, 2, 3, \ldots$, we have

$$\frac{1}{5^{k-1/2}} \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{2k}} = \frac{(-1)^k}{(2k-1)!} \sum_{n=1}^{\infty} \frac{(-1)^n (n-k+1)(n-k+2)\cdots(n+k-1)}{F_{2n}}.$$

***Theorem 4:*** For $k = 1, 2, 3, \ldots$, we have

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^{2k}} = \frac{1}{(2k-1)!\sqrt{5}} \sum_{n=1}^{\infty} \frac{(n-k+1)(n-k+2)\cdots(n+k-1)}{F_{2n}}.$$

Theorems 1 and 2 are corollaries of the following Theorem 5. We note that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. So, if we let $q := q^2$ in Theorem 5, then put $q = \beta$, we have Theorem 1. Similarly, setting $q = \beta$ in Theorem 5 gives Theorem 2.

***Theorem 5:*** For $|q| < 1$ and $k = 1, 2, 3, \ldots$, we have

$$\sum_{n=1}^{\infty} \frac{q^{(2k+1)(2n-1)}}{(1-q^{4n-2})^{2k+1}} = \frac{1}{(2k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1)\cdots(n+k-1)q^{2n-1}}{1-q^{4n-2}}.$$

Theorems 3 and 4 are corollaries of the following theorem.

***Theorem 6:*** For $|q| < 1$ and $k = 1, 2, 3, \ldots$, we have

$$\sum_{n=1}^{\infty} \frac{q^{k(2n-1)}}{(1-q^{2n-1})^{2k}} = \frac{1}{(2k-1)!} \sum_{n=1}^{\infty} \frac{(n-k+1)(n-k+2)\cdots(n+k-1)q^{n}}{1-q^{2n}}.$$

To derive Theorem 3 from Theorem 6, we let $q := -q$. Then $q := q^2$, and we set $q = \beta$ where $\beta = (1-\sqrt{5})/2$. Theorem 4 follows similarly by setting $q := q^2$ then $q = \beta$.

Theorems 5 and 6 are proved in a similar way; therefore, we present only the proof of Theorem 5.

***Proof of Theorem 5:*** For $|q| < 1$ and $k = 1, 2, 3, \ldots$, we have, by the binomial theorem,

$$\sum_{n=1}^{\infty} \frac{q^{(2k+1)(2n-1)}}{(1-q^{4n-2})^{2k+1}} = \sum_{n=1}^{\infty} q^{(2k+1)(2n-1)} \sum_{m=1}^{\infty} \frac{m(m+1)\cdots(m+2k-1)}{(2k)!} q^{(4n-2)(m-1)}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m(m+1)\cdots(m+2k-1)}{(2k)!} q^{(2m+2k-1)(2n-1)}$$

with $m := m-k$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(m-k)(m-k+1)\cdots(m+k-1)}{(2k)!} q^{(2m-1)(2n-1)}$$

which, on interchanging the order of summation,

$$= \frac{1}{(2k)!} \sum_{m=1}^{\infty} \frac{(m-k)(m-k+1)\cdots(m+k-1)q^{2m-1}}{1-q^{4m-2}}.$$

This completes the proof of Theorem 5 and, hence, that of Theorems 1 and 2.

In a similar way to Theorems 1-4, we can demonstrate the following results:

$$\sum_{n=1}^{\infty} \frac{n}{F_{2n}^3} = \frac{5\sqrt{5}}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{L_{2n-1}^2}, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{F_{2n}^3} = \frac{\sqrt{5}}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{F_{2n-1}^2}, \tag{6}$$

and

$$\sum_{n=1}^{\infty}\frac{n}{F_{2n}^4}=\frac{5}{6}\sum_{n=1}^{\infty}\frac{n(n^2-1)}{F_{2n}^2}. \tag{7}$$

The above results are special cases of the following theorems.

***Theorem 7:*** For $k=0, 1, 2, 3, \ldots$, we have

$$\frac{1}{5^{k+\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{n}{F_{2n}^{2k+1}}=\frac{1}{(2k)!}\sum_{n=1}^{\infty}\frac{(n-k)(n-k+1)\cdots(n+k-1)}{L_{2n-1}^2},$$

where $(n-k)(n-k+1)\cdots(n+k-1)$ is taken to be 1 when $k=0$.

***Theorem 8:*** For $k=0, 1, 2, 3, \ldots$, we have

$$\frac{1}{5^{k-\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{F_{2n}^{2k+1}}=\frac{1}{(2k)!}\sum_{n=1}^{\infty}\frac{(n-k)(n-k+1)\cdots(n+k-1)}{F_{2n-1}^2}$$

where $(n-k)(n-k+1)\cdots(n+k-1)$ is taken to be 1 when $k=0$.

***Theorem 9:*** For $k=1, 2, 3, \ldots$, we have

$$\frac{1}{5^{k-1}}\sum_{n=1}^{\infty}\frac{n}{F_{2n}^{2k}}=\frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\frac{(n-k+1)(n-k+2)\cdots(n+k-1)}{F_{2n}^2}.$$

Theorems 7-9 are corollaries of Theorems 10 and 11 below.

***Theorem 10:*** For $|q|<1$ and $k=0,1,2,3,\ldots$, we have

$$\sum_{n=1}^{\infty}\frac{nq^{n(2k+1)}}{(1-q^{2n})^{2k+1}}=\frac{1}{(2k)!}\sum_{n=1}^{\infty}\frac{(n-k)(n-k+1)\cdots(n+k-1)q^{2n-1}}{(1-q^{2n-1})^2},$$

where $(n-k)(n-k+1)\cdots(n+k-1)$ is taken to be 1 when $k=0$.

***Theorem 11:*** For $|q|<1$ and $k=1,2,3,\ldots$, we have

$$\sum_{n=1}^{\infty}\frac{nq^{2nk}}{(1-q^{2n})^{2k}}=\frac{1}{(2k-1)!}\sum_{n=1}^{\infty}\frac{(n-k+1)(n-k+2)\cdots(n+k-1)q^{2n}}{(1-q^{2n})^2}.$$

As with Theorems 5 and 6, the proofs of Theorems 10 and 11 are very similar; thus, we present only the proof of Theorem 11.

***Proof of Theorem 11:*** For $|q|<1$ and $k=1,2,3,\ldots$, we have, by the binomial theorem,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{nq^{2nk}}{(1-q^{2n})^{2k}}&=\sum_{n=1}^{\infty}nq^{2nk}\sum_{m=1}^{\infty}\frac{m(m+1)\cdots(m+2k-2)}{(2k-1)!}q^{2n(m-1)}\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m(m+1)\cdots(m+2k-2)}{(2k-1)!}nq^{n(2m+2k-2)}\end{aligned}$$

with $m:=m-k+1$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-k+1)(m-k+2)\cdots(m+k-1)}{(2k-1)!}nq^{2mn}$$

which, on interchanging the order of summation and summing $\sum_{n=1}^{\infty} nq^{2mn}$,

$$=\frac{1}{(2k-1)!}\sum_{m=1}^{\infty}\frac{(m-k+1)(m-k+2)\cdots(m+k-1)q^{2m}}{(1-q^{2m})^2}.$$

This completes the proof of Theorem 11.

Theorem 7 follows by letting $q:=q^2$, then $q=\beta$ in Theorem 10. Theorem 8 follows by letting $q:=-q$, then $q=\beta$ in Theorem 10, and Theorem 9 follows by letting $q:=q^2$, then $q=\beta$ in Theorem 11.

## REFERENCE

1. J. M. Borwein & P. B. Borwein. *Pi and the AGM.* New York: John Wiley & Sons, 1987.

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