ON SOME PROPERTIES OF FIBONACCI DIAGONALS IN PASCAL'S TRIANGLE

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1. INTRODUCTION

Although it has been studied extensively, Pascal's triangle remains fascinating to explore and there always seems to be some new aspects that are revealed by looking at it closely. In this paper we shall examine a few nice properties of the so-called *Fibonacci diagonals*, that is, those slant lines whose entries sum to consecutive terms of the Fibonacci sequence. We adopt throughout our text the convention that the n^{th} Fibonacci diagonal is the one that contains the binomial coefficients

$$\binom{n-1}{0}\binom{n-2}{1}\binom{n-3}{2}\cdots$$

With that notation, the first diagonal contains only $\binom{0}{0}$, the second one contains only $\binom{1}{0}$, the third one contains $\binom{2}{0}$ and $\binom{1}{1}$, and so on. Addition of the terms of the n^{th} Fibonacci diagonal gives the n^{th} term of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots .$$

For instance, the terms of the 10^{th} Fibonacci diagonal sum to

$$\binom{9}{0}+\binom{8}{1}+\binom{7}{2}+\binom{6}{3}+\binom{5}{4}=1+8+21+20+5=55.$$

We shall also be interested in the corresponding diagonals in *Pascal's triangle mod* 2, that is, the triangle in which the entry $\binom{n}{k}$ is replaced by $\left|{n \atop k}\right|$, its residue mod 2.

Throughout our discussion, it will be convenient to consider the rows or diagonals of Pascal's triangle as vectors with integer components. For instance, the n^{th} horizontal row, $n \geq 0$, will be seen as the vector $\vec{X}_n$ in $\mathbf{Z}^{n+1}$ defined by

$$\vec{X}_n = \left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}\right).$$

Various well-known operations on these rows or diagonals can be seen as the scalar product of such vectors with $\vec{B}_n = (b^n, b^{n-1}, \dots, b, 1) \in \mathbf{Z}^{n+1}$ for some $b \in \mathbf{N}$. Let us give some examples involving the above $\vec{X}_n$. We shall use the notation $\pi_b \vec{X}_n$ to designate the scalar product

$$\vec{X}_n \cdot \vec{B}_n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k}$$

(this notation is motivated by the fact that in some sense the vector $\vec{X}_n$ is being "projected" on the powers of b).

By the Binomial Theorem,

$$\pi_b \vec{X}_n = (b+1)^n. \tag{$*$}$$

In particular, for $b=1$, one gets $\pi_1 \vec{X}_n = 2^n$, i.e., the terms of the n^{th} row of Pascal's triangle sum to 2^n. And for $b=10$, one gets $\pi_{10}\vec{X}_n = (11)^n$. This last equality can be interpreted as follows (see Gardner [1]): when the entries of the rows of Pascal's triangle are considered as the values of a place-value, base-ten numeral, the numbers obtained are the successive powers of 11. We could of course have a similar interpretation by replacing base-ten numeral by base-b numeral and then the powers of 11 by the powers of $(b+1)$.

Note that $(*)$ can be rewritten as

$$\pi_b \vec{X}_n = \pi_{b+1} \vec{1}_n$$

where $\vec{1}_n = (1, 0, 0, \ldots, 0) \in \mathbf{Z}^{n+1}$, with a projection appearing on both sides of the equality sign, but with different bases. Such a "change of base" phenomenon will be encountered again in section 2.

A similar discussion can also be undertaken considering the rows of Pascal's triangle mod 2. The n^{th} row will now be interpreted as the vector

$$\vec{Y}_n = \left(\left|\begin{matrix} n \\ 0 \end{matrix}\right|, \left|\begin{matrix} n \\ 1 \end{matrix}\right|, \left|\begin{matrix} n \\ 2 \end{matrix}\right|, \ldots, \left|\begin{matrix} n \\ n \end{matrix}\right| \right)$$

in $\mathbf{Z}^{n+1}$ with components 0's and 1's. It was shown by Glaisher [2] that the projection

$$\pi_1 \vec{Y}_n = \sum_{k=0}^{n} \left|\begin{matrix} n \\ k \end{matrix}\right|,$$

i.e., the number of odd binomial coefficients $\binom{n}{k}$ for a given n, is again a power of 2, namely $2^{\#(n)}$, where $\#(n)$ represents the number of 1's in the base-two representation of n. For instance, the 5^{th} row vector is $\vec{Y}_5 = (1, 1, 0, 0, 1, 1)$ so that $\pi_1 \vec{Y}_5 = 4 = 2^{\#(5)}$, which corresponds to the fact that 5 is written as 101 in base two with the digit 1 appearing twice. When $b=2$, the projection

$$\pi_2 \vec{Y}_n = \sum_{k=0}^{n} \left|\begin{matrix} n \\ k \end{matrix}\right| 2^{n-k}$$

gives *Gould's numbers*. These numbers were introduced in Gould [3], where a recursion formula was given for them and a relationship with Fermat's primes was obtained (see also Hodgson [4] for details).

We shall be concerned in this paper with the study of analogous results obtained when Fibonacci diagonals are considered instead of horizontal rows, both in Pascal's triangle and in Pascal's triangle mod 2.

2. FIBONACCI DIAGONALS IN THE STANDARD PASCAL TRIANGLE

Recall that Fibonacci diagonals are numbered starting with $n=1$. For further reference, we list the first twelve vectors thus obtained:

$\vec{S}_1 = (1)$	$\vec{S}_7 = (1, 5, 6, 1)$
$\vec{S}_2 = (1)$	$\vec{S}_8 = (1, 6, 10, 4)$
$\vec{S}_3 = (1, 1)$	$\vec{S}_9 = (1, 7, 15, 10, 1)$
$\vec{S}_4 = (1, 2)$	$\vec{S}_{10} = (1, 8, 21, 20, 5)$
$\vec{S}_5 = (1, 3, 1)$	$\vec{S}_{11} = (1, 9, 28, 35, 15, 1)$
$\vec{S}_6 = (1, 4, 3)$	$\vec{S}_{12} = (1, 10, 36, 56, 35, 6)$

Clearly $\pi_1\vec{S}_n$ gives the n^{th} term of the Fibonacci sequence. We now study the projections $\pi_b\vec{S}_n$ for $b \in \mathbf{N}$.

We first note that, for all $n \geq 1$, $\vec{S}_{2n-1}$ and $\vec{S}_{2n}$ are both vectors in $\mathbf{Z}^n$. The following notation will be convenient in the sequel. For $\vec{S}_n = (a_1, a_2, a_3, \ldots)$, we say that $i_n\vec{S}_n = (a_1, 0, -a_2, 0, a_3, \ldots)$ is the *image* of $\vec{S}_n$ in $\mathbf{Z}^n$ and that $i_{n+1}\vec{S}_n = (0, a_1, 0, -a_2, 0, a_3, \ldots)$ is the *image* of $\vec{S}_n$ in $\mathbf{Z}^{n+1}$ (note that these image vectors are obtained by assigning in turn + and – signs to the components of $\vec{S}_n$ and then inserting 0's in between those entries).

Before stating the general result, it is instructive to look at a few examples. Let us first consider the vector $\vec{S}_8 = (1, 6, 10, 4) \in \mathbf{Z}^4$; clearly $\pi_{10}\vec{S}_8 = 1\cdot 10^3 + 6\cdot 10^2 + 10\cdot 10^1 + 4\cdot 10^0 = 1704$. It can also be checked that 1704 can be given by a simple expression involving only the entries of $\vec{S}_4 = (1, 2)$, namely, $1704 = 1\cdot 12^3 - 2\cdot 12^1$; we can thus write $\pi_{10}\vec{S}_8 = \pi_{12}i_4\vec{S}_4$, where $i_4\vec{S}_4 = (1, 0, -2, 0)$.

For $\vec{S}_{10} = (1, 8, 21, 20, 5)$, we find $\pi_{10}\vec{S}_{10} = 20305$; since $i_5\vec{S}_5 = (1, 0, -3, 0, 1)$, we obtain similarly

$$\pi_{12}i_5\vec{S}_5 = 1\cdot 12^4 - 3\cdot 12^2 + 1\cdot 12^0 = 20736 - 432 + 1 = 20305 = \pi_{10}\vec{S}_{10}.$$

On the other hand, for $\vec{S}_9 = (1, 7, 15, 10, 1)$, we have $\pi_{10}\vec{S}_9 = 18601$; introducing the two image vectors $i_5\vec{S}_5 = (1, 0, -3, 0, 1)$ and $i_5\vec{S}_4 = (0, 1, 0, -2, 0)$, it is easily checked that

$$\pi_{12}i_5\vec{S}_5 - \pi_{12}i_5\vec{S}_4 = 18601 = \pi_{10}\vec{S}_9.$$

The use of base $b = 10$ was by no means essential in the above examples, as we shall now show.

Theorem:

a) $\pi_b\vec{S}_{2n} = \pi_{b+2}i_n\vec{S}_n, \quad n \geq 1.$

b) $\pi_b\vec{S}_{2n-1} = \pi_{b+2}i_n\vec{S}_n - \pi_{b+2}i_n\vec{S}_{n-1}, \quad n \geq 2.$

Proof: Using the basic recursion formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

it is easily seen that

$$\pi_b\vec{S}_{2n-1} + \pi_b\vec{S}_{2n-2} = \pi_b\vec{S}_{2n}, \quad \text{that is,} \quad \pi_b\vec{S}_{2n-1} = \pi_b\vec{S}_{2n} - \pi_b\vec{S}_{2n-2}.$$

It is thus sufficient to prove a), since b) then follows at once.

Proof of a): Let us expand both sides of the required equality. One must thus establish that

$$\sum_{k=0}^{n-1}\binom{2n-1-k}{k}b^{n-1-k}=\sum_{t=0}^{\left[\frac{n-1}{2}\right]}(-1)^t\binom{n-1-t}{t}(b+2)^{n-1-2t},$$

where $[x]$ denotes the integer part of x. This can of course be done using the techniques of generating functions. We prefer, however, to give a proof based on a common combinatorial interpretation of both sides.

We first use the Binomial Theorem to replace the last factor in the right-hand side of the above, thus getting

$$\sum_{k=0}^{n-1}\binom{2n-1-k}{k}b^{n-1-k}=\sum_{t=0}^{\left[\frac{n-1}{2}\right]}(-1)^t\binom{n-1-t}{t}\left[\sum_{u=0}^{n-1-2t}\binom{n-1-2t}{u}b^{n-1-2t-u}2^u\right].$$

We now need to expand the right-hand side of this inequality as a polynomial in b and then compare the coefficients of the powers of b with those occurring on the left-hand side. For a fixed k, we are thus interested in values of t and u such that $u=k-2t$, since only these terms will contribute to the coefficient of b^{n-1-k}. One is then lead to prove that

$$\binom{2n-1-k}{k}=\sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j2^{k-2j}\binom{n-1-j}{j}\binom{n-1-2j}{k-2j}$$

or, equivalently, that

$$\binom{2n-1-k}{k}=2^k\cdot\binom{n-1}{k}-\sum_{j=1}^{\left[\frac{k}{2}\right]}(-1)^{j+1}2^{k-2j}\binom{n-1-j}{j}\binom{n-1-2j}{k-2j} \qquad (**)$$

for $k\le n-1$.

It is easily verified, for instance by induction on k, that $\binom{m+1-k}{k}$ can be interpreted as the number of ways of selecting k integers among 1, 2, 3, ..., m in such a way that no two of them are consecutive. The left-hand side of $(**)$ can thus be seen as the number of ways of picking k integers among 1, 2, 3, ..., $2n-2$, no two of them being consecutive.

We want to show, of course, that the right-hand side of $(**)$ counts exactly the same number. Let us first observe that the first term, $2^k\cdot\binom{n-1}{k}$, can be seen as the number of ways of picking k integers among 1, 2, 3, ..., $2n-2$ by the following two-step procedure:

Step 1: Select k pairs of integers of the form $\{2s-1, 2s\}$ among 1, 2, 3, ..., $2n-2$. This can be done in $\binom{n-1}{k}$ ways.

Step 2: Pick an integer in each of the k pairs selected. This can be done in 2^k ways.

While this procedure clearly generates any set of k integers chosen among 1, 2, 3, ..., $2n-2$ in such a way that no two of them are consecutive, it does, however, also allow picking both integers $2i$ and $2i+1$. When this happens, we shall say that *the event A_i has occurred.* Note that, in such a case, the index i can take the values 1, 2, 3, ..., $n-2$. Also, when both events A_{i_1} and A_{i_2} $(i_1\ne i_2)$ occur within a given selection of integers, the indices i_1 and i_2 are not consecutive.

It thus remains to show that the number of elements corresponding to the event $A_1 \cup A_2 \cup \cdots \cup A_{n-2}$ is given exactly by the subtrahend on the right-hand side of $(**)$. Such a proof follows directly from the usual "inclusion-exclusion technique" for counting the elements in a union of events: one first $(j=1)$ adds up the counts in each A_i, one then $(j=2)$ subtracts the counts in each $A_{i_1} \cap A_{i_2}$ $(i_1 < i_2)$, then $(j=3)$ one adds the counts in each $A_{i_1} \cap A_{i_2} \cap A_{i_3}$ $(i_1 < i_2 < i_3)$, etc.

Let us consider, for instance, the case $j = 1$. $\binom{n-1-1}{1}$ is the number of ways of selecting an index i (that is, two integers) so that the event A_i has occurred. In order to complete a choice of k integers, one first selects $k-2$ pairs among the remaining integers [Step 1—this can be done in $\binom{n-1-2}{k-2}$ ways], and then —Step 2—picks one integer from each of these pairs (which can be done in 2^{k-2} ways).

A similar argument applies generally for any $j>1$. One must first note that $\binom{n-1-j}{j}$ is the number of ways of selecting the indices $i_1 < i_2 < \cdots < i_j$ in such a way that no two of them are consecutive ($2j$ integers are thus chosen through this stage). Then, as above, $\binom{n-1-2j}{k-2j}$ counts the number of ways of selecting $k-2j$ pairs among the remaining integers—Step 1—and 2^{k-2j} is the number of ways of performing Step 2.

The theorem is thus proven. □

Taking $b=1$ in the above theorem, we have the following equalities:

a) $\pi_1 \vec{S}_{2n} = \pi_3 i_n \vec{S}_n, \quad n \ge 1.$

b) $\pi_1 \vec{S}_{2n-1} = \pi_3 i_n \vec{S}_n - \pi_3 i_n \vec{S}_{n-1}, \quad n \ge 2.$

Hence, the $(2n)^{\text{th}}$ Fibonacci number can be calculated by using a base-three interpretation of the n^{th} Fibonacci diagonal, whereas the $(2n-1)^{\text{th}}$ Fibonacci number can be calculated via a base-three interpretation of both the n^{th} and the $(n-1)^{\text{th}}$ Fibonacci diagonals. For instance, the 6^{th} Fibonacci number is 8 and it can be obtained from $\vec{S}_3 = (1, 1)$ as $1 \cdot 3^2 - 1 \cdot 3^0$. The 11^{th} Fibonacci number is 89, which can be obtained via the diagonals $\vec{S}_6 = (1, 4, 3)$ and $\vec{S}_5 = (1, 3, 1)$: one has here

$$\pi_3 i_6 \vec{S}_6 = 1 \cdot 3^5 - 4 \cdot 3^3 + 3 \cdot 3^1 = 144$$

and

$$\pi_3 i_6 \vec{S}_5 = 1 \cdot 3^4 - 3 \cdot 3^2 + 1 \cdot 3^0 = 55.$$

3. FIBONACCI DIAGONALS IN PASCAL'S TRIANGLE MOD 2

The Theorem of section 2 tells us how certain computations regarding Fibonacci diagonals can be "lifted" to computations done just half-way down Pascal's triangle. Such a theorem is in the same spirit as the results presented in Hodgson [4] with respect to Pascal's triangle mod 2. We now briefly recall these results.

Let us denote by $\vec{T}_n, n \ge 1$, the vector representing the n^{th} Fibonacci diagonal mod 2. These diagonals have already been studied in Hodgson [4] where numbers H_n, analogous to Gould's numbers, have been introduced. In our notation, we have $H_n = \pi_2 \vec{T}_n$. The following calculation rules for H_n were proven in Hodgson [4] (see Proposition 6.1 therein):

$$H_{2^h} = 2^{2^{h-1}-1}. \tag{i}$$

$$H_{2^h+u} = H_u \cdot 2^{2^{h-1}} + H_{2^h-u} \text{ for } 1 \le u < 2^h. \tag{ii}$$

(The reader should be aware that the slightly different form of those rules in [4] is due to the numbering of diagonals there starting with $n = 0$.) The proof of these recursion formulas is essentially based on an algebraic translation of the "geometry" of Pascal's triangle mod 2, that is, the very interesting way in which the 0's and the 1's are distributed (the reader should write down the first n rows of that triangle and observe the nice pattern obtained).

We now end this paper by describing techniques that allow the computation of both $\pi_2\vec{T}_n$ and $\pi_1\vec{T}_n$ in a most direct fashion. In opposition to the above formulas that relate the value of a certain H_n to powers of 2 and previous H_i's, the procedures below give the value of both $H_n = \pi_2\vec{T}_n$ and $\pi_1\vec{T}_n$ by working directly on the index n. Figures 2 and 3 illustrate the simplicity of these methods, whose validity is a consequence of the following discussion.

For convenience, let us introduce the notation t_n to represent the base-two representation of H_n. (Note that t_n can be simply seen as the vector $\vec{T}_n$ with the commas removed.) Formulas (i) and (ii) now become

$$t_{2^h} = 1000\ldots0 \quad (2^{h-1} - 1 \text{ zeros}), \tag{i'}$$

$$t_{2^h+u} = t_u 000\ldots0 t_{2^h-u} \quad \text{for } 1 \le u < 2^h, \tag{ii'}$$

where the number of intermediate zeros is such that the string $000\ldots0t_{2^h-u}$ is made of exactly 2^{h-1} digits. As an example, let us compute t_{29}. Since it is trivially verified that $t_3 = 11$, we thus have

$$\begin{aligned} t_{29} &= t_{13}000000t_3 \\ &= t_5 00 t_3 00000011 \\ &= t_1 t_3 001100000011 \\ &= 111001100000011 \end{aligned}$$

[the number of intermediate zeros introduced at each computation step follows from (ii')].

The preceding calculations can also be conveniently displayed as in the tableau of Figure 1. In general, given $n = 2^h + u$, we shall need a tableau made of h rows, each one containing 2^h positions to be ultimately filled at the last step of the procedure. Rows are indexed by decreasing powers of two that serve to split each number appearing on the preceding row. At the row of index 2^k, any number (from row 2^{k+1}) of the form $2^k + v$ becomes split into v and $2^k - v$, while any number $w \le 2^k$ splits into 0 and w. This procedure may be better grasped by displaying the entries as in the tree diagram given in Figure 2. (For odd n, this algorithm directly gives t_n at its last step of computation. However, because of parity considerations, the last row will, for even n, always contain 0's and 2's: we note that t_n can then be obtained by merely replacing each digit 2 by a 1.)

We finally present a technique for the computation of $\pi_1\vec{T}_n$ (compare with Glaisher's rule for the calculation of $\pi_1\vec{Y}_n$ mentioned in the Introduction). Note that we are now interested solely in the total number of 1's, and no longer in their exact position. All amounts to finding how one can build n using only powers of two—or, if one prefers, to what extent n is "far" from being itself a power of two. For this purpose, we introduce a notion of *weight*. The diagram of Figure 3 (for $n = 29$) helps to clarify the discussion. Let us read that diagram from the bottom up. Powers of two (here, 16 and 32) are considered to be of weight 1. Then 24, being halfway between powers of two, is of weight 2 (= 1 + 1). Since 28 is halfway between 24 and a power of two, it is given weight $2+1=3$. Continuing in this manner, 29 receives a weight of $3+4=7$: this weight is also the value of $\pi_1\vec{T}_{29}$, the total number of 1's appearing in t_{29}. (It is usually more convenient to consider Figure 3 as being built from the top down, with the weights being incorporated into the diagram at the end of the process.) The general validity of this procedure follows from recursive applications of formulas (i′) and (ii′) above.

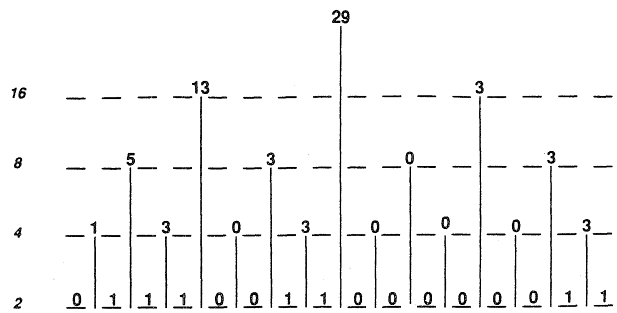

Figure 1

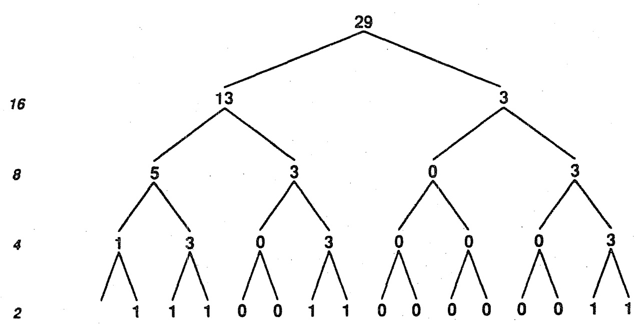

Figure 2

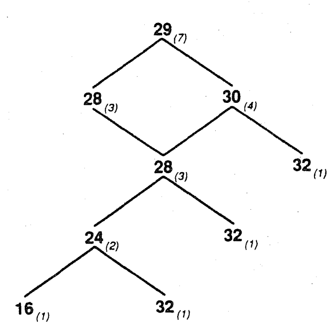

Figure 3

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