EXTENSIONS TO THE GCD STAR OF DAVID THEOREM

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1. INTRODUCTION

The GCD Star of David Theorem asserts that

$$\gcd\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\} = \gcd\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\}. \tag{1.1}$$

It was first conjectured by H. W. Gould [2] in 1972.

It has been proven and/or generalized by Hillman & Hoggatt [3, 4], Strauss [11], Hitotumatu & Sato [5], Ando [1], Singmaster [10], and Long & Ando [7, 8].

In this paper we will show some figures other than the hexagon described by the binomial coefficients in (1.1) that also have a gcd property. We also give a method whereby a new polygon with a gcd property can be constructed from known polygons with that property.

2. TERMINOLOGY

By a polygonal figure P in Pascal's triangle, we mean a simple closed polygonal curve whose vertices are entries of Pascal's triangle. We also use the same symbol to represent the set of entries on the curve. The six binomial coefficients in (1.1) form a hexagon with $\binom{n}{k}$ at its center. This hexagon will be called a fundamental hexagon.

Following Long [6], we say P is tiled by fundamental hexagons if P is "covered" by a set $\mathfrak{F}$ of fundamental hexagons F in such a way that

(1) The vertices of each F in $\mathfrak{F}$ are coefficients in P or interior of P.
(2) Each boundary coefficient of P is a vertex of precisely one F in $\mathfrak{F}$.
(3) Each interior coefficient of P is interior to some F in $\mathfrak{F}$ or is a vertex shared by precisely two elements in $\mathfrak{F}$.

If, in addition, the tiling has the property

(4) For all F_1 and F_2 in $\mathfrak{F}$, F_1 and F_2 have at most one vertex in common.

we say P has a *restricted tiling*. The three polygons in Figure 1 illustrate the three possibilities. The upper left figure has no tiling. The bottom figure has a restricted tiling.

Let P be some configuration of binomial coefficients in Pascal's triangle. Suppose $P = X \cup Y$. If $\gcd X = \gcd Y$ independent of the placement of P in Pascal's triangle, then P is said to have the gcd property with respect to X and Y. The fundamental hexagon has this property with respect to the two sets of three coefficients on the alternate vertices of the hexagon.

An $m \times n \times k$ hexagon is a hexagon oriented along the rows and main diagonals of Pascal's triangle with $m, n, k, m, n,$ and k entries per side starting with m entries on top and going clockwise around the hexagon.

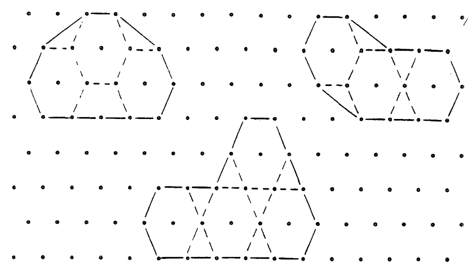

FIGURE 1. Some Polygons with Their Tilings

3. THE GENERAL METHOD

If we know that polygons P_1 and P_2 have the gcd property with respect to certain sets, we can construct a larger polygon with a gcd property by using the following theorem.

Theorem 1: Let P_1 and P_2 be two configurations. Suppose P_1 has the gcd property with respect to X and $Y \cup S$, and suppose P_2 has the gcd property with respect to U and $V \cup S$. Then $P = X \cup V \cup Y \cup U$ has the gcd property with respect to $X \cup V$ and $U \cup Y$.

Proof: gcd $X \cup V = \gcd(Y \cup S) \cup V = \gcd Y \cup (S \cup V) = \gcd Y \cup U$.

Some figures satisfy the hypotheses of Theorem 1 in a very obvious way. We have the following corollary to Theorem 1.

Corollary 1: Let P be a $2 \times 2 \times 2n$, $2 \times 2n \times 2$, or $2n \times 2 \times 2$ hexagon in Pascal's triangle. Label the elements along the boundary $a_1, a_2, a_3, \ldots, a_{4n+2}$ in sequence. Let $X = \{a_1, a_3, \ldots, a_{4n+1}\}$ and $Y = \{a_2, a_4, \ldots, a_{4n+2}\}$. Then $\gcd X = \gcd Y$.

Each of the hexagons described above admits a restricted tiling by n fundamental hexagons. The corollary is easily proved by induction on n with Theorem 1 providing the inductive step.

4. OTHER POLYGONS WITH THE GCD PROPERTY

In what follows, we will label polygons in the following manner unless otherwise noted. The left-most vertex on the top row will be labeled a_1. Then as we travel clockwise along the boundary of the hexagon we label the coefficients $a_2, a_3, a_4, \ldots$.

We will show that any polygon with a restricted tiling of fewer than five fundamental hexagons has the gcd property with respect to the sets $\{a_i \mid i \text{ odd}\}$ and $\{a_i \mid i \text{ even}\}$.

First, a polygon P that has such a tiling by one fundamental hexagon must be a fundamental hexagon. The GCD Star of David Theorem gives the desired result here.

Suppose P has a restricted tiling by two fundamental hexagons. Then P is either the disjoint union of two fundamental hexagons or is a $2 \times 2 \times 4$, $2 \times 4 \times 2$, or a $4 \times 2 \times 2$ hexagon. In the former case, each fundamental hexagon has the required gcd property and thus their disjoint union will also. The hexagons described in the latter case were shown in Corollary 1 to have the desired gcd property.

At this point, we drop from consideration the polygons that are comprised of two or more components, since their gcd properties are inherited from the separate components.

Now, let P have a restricted tiling by three fundamental hexagons. There are two cases. Either some fundamental hexagon intersects only one of the other fundamental hexagons or each fundamental hexagon intersects both of the remaining two fundamental hexagons.

The polygons in the first case have the desired gcd property as a result of Theorem 1; the polygons described in the second case are either $2\times 4\times 2\times 4\times 2\times 4$ or $4\times 2\times 4\times 2\times 4\times 2$ hexagons.

The former is shown with its restricted tiling in Figure 2. We will show that each of the hexagons has the gcd property with respect to the sets $\{a_1, a_3, \ldots, a_{11}\}$ and $\{a_2, a_4, \ldots, a_{12}\}$.

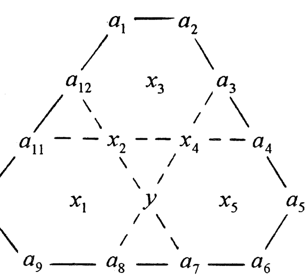

FIGURE 2. A Diamond Formed by Three Fundamental Hexagons

We start with the hexagon in Figure 2. First, we prove the following lemma.

Lemma 1: With the notation of Figure 2,

$$\gcd\{y, a_1, a_3, \ldots, a_{11}\} = \gcd\{y, a_2, a_4, \ldots, a_{12}\}. \tag{4.1}$$

Proof: Applying Theorem 1 for

$$X = \{a_2, a_4, a_6, a_{12}, y\}, \quad Y = \{a_1, a_3, a_5, a_7\},$$
$$U = \{y, a_9, a_{11}\}, \quad V = \{a_8, a_{10}\}, \quad S = \{x_2\},$$

(4.1) holds as claimed.

We show that the element y is superfluous in this lemma.

Theorem 2: With the notation of Figure 2,

$$\gcd\{a_1, a_3, \ldots, a_{11}\} = \gcd\{a_2, a_4, \ldots, a_{12}\}. \tag{4.2}$$

Proof: We will make use of the notation $v_p(n) = e$. By this we mean that $p^e \| n$. We will drop the subscript p when no confusion arises about which base p is to be used. Also, the notation $v_p(X) = e$ will imply that $p^e \| \gcd X$.

Now suppose that $h = \gcd\{a_1, a_3, \ldots, a_{11}\}$ and $g = \gcd\{a_2, a_4, \ldots, a_{12}\}$ and that $h > g$. Then there exists a prime p for which $v_p(h) = e > v_p(g)$.

If $v(a_6) \geq e$, then $v(x_5) = v(a_6 - a_5) \geq e$, which implies $v(y) = v(a_7 - x_5) \geq e$. Then, from Lemma 1, $v(\{y, a_2, a_4, \ldots, a_{12}\}) \geq e$. Thus, $v(g) \geq e$, and this is a contradiction. Similarly, if $v(a_{10}) \geq e$ or $v(a_2) \geq e$, then using Lemma 1 with y replaced by x_2 or x_4, respectively, we have the same conclusion. Therefore, $v(a_2) < e$, $v(a_6) < e$, and $v(a_{10}) < e$.

Now, $a_1a_5a_9 = a_2a_6a_{10}$. (See [9].) We know that $v(a_1a_5a_9) \geq 3e$; thus, $v(a_2a_6a_{10}) \geq 3e$, and this is a contradiction. Hence, $h \leq g$. Similarly, $g \leq h$. Therefore, $g = h$ and the theorem is proved.

To show that the $4 \times 2 \times 4 \times 2 \times 4 \times 2$ hexagon has the gcd property with respect to the same two sets, we first prove a lemma. To help with this lemma, we label a polygon P a little differently. Assume one of the boundaries of P falls along a row or main diagonal of Pascal's triangle and that the boundary consists of four or more consecutive binomial coefficients. We wish to adjoin a fundamental hexagon H to P to the right of the diagonal $n-k=c$, to the left of the diagonal $k=c$, or below the row $n=c$. This is illustrated in Figure 3. We label P so that a_1, a_2, a_3, and a_4 are labeled as in Figure 3 and then continue around P in the direction indicated labeling the coefficients $a_5, a_6, \ldots, a_{2n}$.

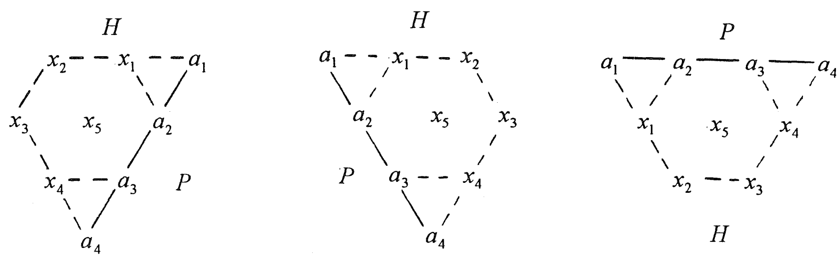

FIGURE 3. Adding a Fundamental Hexagon to a Polygon

Using this convention, we are now prepared to prove the following lemma.

Lemma 2: Let P be a polygon in Pascal's triangle as labeled above. Suppose P has the gcd property with respect to $S = \{a_{2i-1} \mid i = 1, 2, \ldots, n\}$ and $T = \{a_{2i} \mid i = 1, 2, \ldots, n\}$; that is, $\gcd\{a_i \mid i \text{ odd}\} = \gcd\{a_i \mid i \text{ even}\}$. Let H be the fundamental hexagon $\{a_2, a_3, x_1, x_2, x_3, x_4\}$ as in Figure 3. Then the polygon formed by $\{a_1, a_4, a_5, a_6, \ldots, a_{2n}, x_1, \ldots, x_4\}$ has the gcd property with respect to $X = \{a_1, a_5, a_7, a_9, \ldots, a_{2n-1}, x_2, x_4\}$ and $Y = \{a_4, a_6, \ldots, a_{2n}, x_1, x_3\}$.

Proof: Suppose p is a prime for which $e = v_p(X) > v_p(Y)$. Then we have $s = v(a_3) < e$.

If $t = v(a_2) \geq e$, then $v(\{x_1, x_3, a_3\}) = v(\{a_2, x_2, x_4\}) \geq e$. This is a contradiction. Hence, we have $v(a_2) < e$.

From this and $v(X) = e$, we have

$$v(x_1) = v(a_2) = v(x_5) = v(x_3) = t < e.$$

Thus, $v(a_2x_2x_4) \geq 2e + t$ and $v(x_1a_3x_3) = 2t + s$. We have $2t + s \geq 2e + t$, since $a_2x_2x_4 = x_1a_3x_3$. This reduces to $t + s \geq 2e$, which is a contradiction.

Hence, $v(X) \leq v(Y)$ for any prime p. The argument to show that $v(Y) \leq v(X)$ is similar, and is omitted here. From this, it follows that $\gcd X = \gcd Y$.

Theorem 3: Using the notation given in Figure 4 below,

$$\gcd\{a_1, a_3, \ldots, a_{11}\} = \gcd\{a_2, a_4, \ldots, a_{12}\}. \tag{4.3}$$

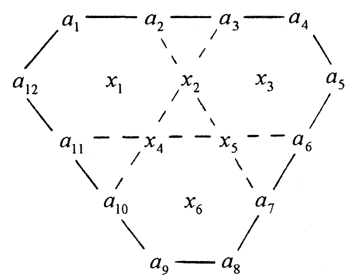

FIGURE 4. The $4 \times 2 \times 4 \times 2 \times 4 \times 2$ Hexagon

Proof: The $4 \times 2 \times 2$ hexagon at the top has the gcd property with respect to the sets $\{a_1, a_3, a_5, x_5, a_{11}\}$ and $\{a_2, a_4, a_6, x_4, a_{12}\}$. Lemma 2 applies with P as the $4 \times 2 \times 2$ hexagon and H as the fundamental hexagon centered at x_6. This gives the desired result in (4.3).

Now let P be any polygon that has a restricted tiling of four fundamental hexagons. If there is a fundamental hexagon in the tiling that intersects only one of the other fundamental hexagons in the tiling, the polygon P will have the desired gcd property. Theorem 1 would give the result using the fundamental hexagon as one polygon and the other component as the second polygon.

Suppose then that each fundamental hexagon intersects at least two of the other fundamental hexagons. The only possibilities are shown with their tilings in Figure 5. They are the $2 \times 4 \times 4$, $4 \times 2 \times 4$, and $4 \times 4 \times 2$ hexagons.

Each of these hexagons has the desired gcd property. Each of these three cases follows from Theorem 2 and Lemma 2. The fundamental hexagon has been placed on the upper right, bottom, and upper left, respectively, to obtain the three hexagons.

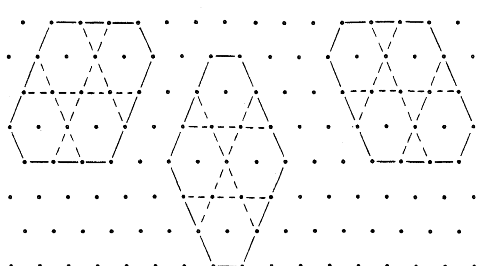

FIGURE 5. Hexagons Tiled by Four Fundamental Hexagons

Therefore, we have the following theorem.

Theorem 4: Lete P be any polygon in Pascal's triangle that admits a restricted tiling of four or fewer fundamental hexagons. Then P has the gcd property with respect to the sets $\{a_i \mid i \text{ odd}\}$ and $\{a_i \mid i \text{ even}\}$.

This can also be extended to the $4 \times 2 \times 6 \times 2 \times 4 \times 4$, the $4 \times 4 \times 4 \times 2 \times 6 \times 2$, and the $6 \times 2 \times 4 \times 4 \times 4 \times 2$ hexagons using Theorem 4 and Lemma 2.

Consider the polygon in Figure 6 below. The $4 \times 4 \times 4$ hexagon P defined by $\{a_1, x_1, x_2, a_6, a_7, a_8, \ldots, a_{20}\}$ has been shown by Long & Ando [8] to have the gcd property with respect to $\{a_1, x_2, a_7, a_9, \ldots, a_{19}\}$ and $\{x_1, a_6, a_8, \ldots, a_{20}\}$. There is a fundamental hexagon H centered at x_3.

Apply Lemma 2 with this P and H. We see that the polygon of Figure 6 has the gcd property with respect to $\{a_1, a_3, \ldots, a_{19}\}$ and $\{a_2, a_4, \ldots, a_{20}\}$. This polygon has no tiling, restricted or otherwise, of fundamental hexagons.

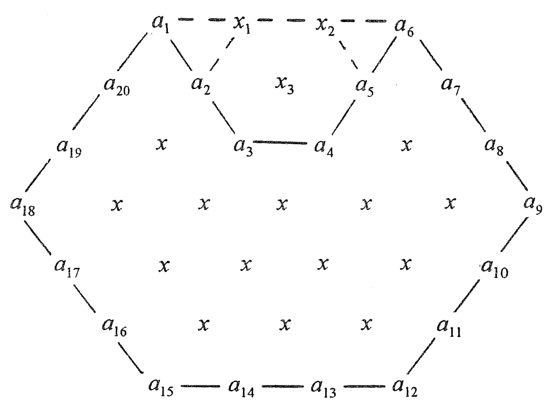

FIGURE 6. A Polygon that Has No Tiling

Consider the polygon of Figure 7, which can be tiled by fundamental hexagons as illustrated. It does not have a restricted tiling. If $X = \{a_1, a_3, a_5, a_7\}$ and $Y = \{a_2, a_4, a_6, a_8\}$, we do not have $\gcd X = \gcd Y$. If we place the octagon so that $a_1 = \binom{22}{11}$, we have $\gcd X = 1292$ and $\gcd Y = 646$.

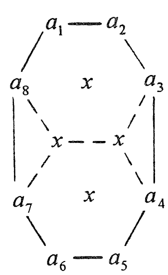

FIGURE 7. An Octagon

We close with the following observation. A polygon possessing a restricted tiling seems to have the desired gcd property. It is by no means a necessary condition, as the polygon in Figure 6 shows. However, possessing a tiling that is not a restricted tiling is not sufficient to guarantee the desired gcd property, as the octagon of Figure 7 shows.

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