THE MULTIPARAMETER NONCENTRAL STIRLING NUMBERS

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1. INTRODUCTION

The Sterling numbers of the first and second kind were introduced by Stirling in 1749 (see [9]). Recently, several generalizations and extensions of the Stirling numbers are given and many combinatorial, probabilistic, and statistical applications are discussed (see [1], [2], [3], [4], and [8]).

In a recent paper [6], Koutras defined $s(n,k;\alpha)$ and $S(n,k;\alpha)$ [we used these symbols instead of $s_\alpha(n,k)$ and $S_\alpha(n,k)$ to avoid ambiguity with Comtet's numbers], the noncentral Stirling numbers of the first and second kind, by

$$(t)_n = \sum_{k=0}^{n} s(n,k;\alpha)\,(t-\alpha)^k, \tag{1.1}$$

$$(t-\alpha)^n = \sum_{k=0}^{n} S(n,k;\alpha)\,(t)_k. \tag{1.2}$$

In this paper we use the following notations:

$$(t/\alpha)_n = \prod_{j=0}^{n-1}(t-\alpha_j),\ (t/\alpha)_0 = 1,\ \text{ and }\ (\alpha_k)_\ell = \prod_{\substack{j=0\\ j\neq k}}^{\ell}(\alpha_k-\alpha_j),\ k\le \ell.$$

Comtet [5] defined $s_\alpha(n,k)$ and $S_\alpha(n,k)$, the generalized Stirling numbers of the first and second kind associated with $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, by

$$(t/\alpha)_n = \sum_{k=0}^{n} s_\alpha(n,k)t^k, \tag{1.3}$$

$$t^n = \sum_{k=0}^{n} S_\alpha(n,k)\,(t/\alpha)_k. \tag{1.4}$$

The main purpose of this paper is to modify the noncentral Stirling numbers of the first and second kind.

In sections 2 and 3 we define $s(n,k;\overline{\alpha})$ and $S(n,k;\overline{\alpha})$, the multiparameter noncentral Stirling numbers of the first and second kind; recurrence relations, generating functions, and explicit forms are obtained.

Some special cases are discussed and a relation between the multiparameter noncentral Stirling numbers and other Stirling numbers are found. Finally, in section 4, some applications are derived.

2. THE MULTIPARAMETER NONCENTRAL STIRLING NUMBERS OF THE FIRST KIND

Definition: Let t be a real number, n a nonnegative integer, and $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ where $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1}$ are real numbers.

We define the multiparameter noncentral Stirling numbers of the first kind, $s(n, k;\ \alpha_0, a_1, \ldots, \alpha_{n-1})$, briefly denoted by $s(n, k; \overline{\alpha})$, with parameters $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, by

$$(t)_n = \sum_{k=0}^{n} s(n, k; \overline{\alpha})\, (t / \alpha)_k, \tag{2.1}$$

where $s(0, 0; \overline{\alpha}) = 1$ and $s(n, k; \overline{\alpha}) = 0$ for $k > n$.

Theorem 2.1: The multiparameter noncentral Stirling numbers of the first kind $s(n, k; \overline{\alpha})$ satisfy the recurrence relation

$$s(n+1, k; \overline{\alpha}) = s(n, k-1; \overline{\alpha}) + (\alpha_k - n)s(n, k; \overline{\alpha}) \quad \text{for } k \geq 1, \tag{2.2}$$

where $s(0, 0; \overline{\alpha}) = 1$ and $s(n, k; \overline{\alpha}) = 0$ for $k > n$ and

$$s(n, 0; \overline{\alpha}) = (\alpha_0 - n + 1)(\alpha_0 - n + 2) \cdots (\alpha_0 - 1)\alpha_0 = (\alpha_0)_n.$$

Proof: Since $(t)_{n+1} = (t)_n[(t - \alpha_k) + (\alpha_k - n)]$, we have

$$\begin{aligned}\sum_{k=0}^{n+1} s(n+1, k; \overline{\alpha})\, (t / \alpha)_k &= (t - \alpha_k)\sum_{k=0}^{n} s(n, k; \overline{\alpha})\, (t / \alpha)_k + (\alpha_k - n)\sum_{k=0}^{n} s(n, k; \overline{\alpha})\, (t / \alpha)_k \\ &= \sum_{k=1}^{n+1} s(n, k-1; \overline{\alpha})\, (t / \alpha)_k + (\alpha_k - n)\sum_{k=0}^{n} s(n, k; \overline{\alpha})\, (t / \alpha)_k.\end{aligned}$$

Equating the coefficients of $(t / \alpha)_k$ on both sides, we get (2.2). For $k = 0$ we get $s(n+1, 0; \overline{\alpha}) = (\alpha_0 - n)s(n, 0; \overline{\alpha})$; therefore, $s(n, 0; \overline{\alpha}) = (\alpha_0)_n$ follows by induction.

Remarks: We discuss the following special cases:

i) If $\alpha_i = \alpha$, $i = 0, 1, \ldots, n-1$, then from (2.2) we have

$$s(n+1, k; \alpha) = s(n, k-1; \alpha) + (\alpha - n)s(n, k; \alpha),$$

where $s(n, k; \alpha)$ denotes the noncentral Stirling numbers of the first kind that is defined by Koutras [6].

ii) If $\alpha_i = 0$, $i = 0, 1, \ldots, n-1$, then we have

$$s(n+1, k) = s(n, k-1) - ns(n, k),$$

where $s(n, k)$ denotes the usual Stirling numbers of the first kind [9].

iii) If $\alpha_i = i$, $i = 0, 1, \ldots, n-1$, then $s(n, k; \overline{\alpha})$ reduces to the C-numbers, where $r = 1$, i.e., $C(n, k, 1)$ (see [3]).

Theorem 2.2: The multiparameter noncentral Stirling numbers of the first kind have the exponential generating function

$$\phi_k(t;\overline{\alpha}) = \sum_{n=k}^{\infty} s(n,k;\overline{\alpha})\frac{t^n}{n!} = \sum_{j=0}^{k}\frac{(1+t)^{\alpha_j}}{(\alpha_j)_k}. \tag{2.3}$$

Proof: Let $\phi_k(t,\overline{\alpha})$ be the exponential generating function of $s(n,k;\overline{\alpha})$, then

$$\begin{aligned} \phi_k(t;\overline{\alpha}) &= \sum_{n=0}^{\infty} s(n,k;\overline{\alpha})\frac{t^n}{n!}, \quad \text{where} \\ \phi_0(t;\overline{\alpha}) &= \sum_{n=0}^{\infty} s(n,0;\overline{\alpha})\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\alpha_0)_n\frac{t^n}{n!} = (1+t)^{\alpha_0}. \end{aligned} \tag{2.4}$$

Differentiating both sides of (2.4) with respect to t, we get

$$\phi_k'(t;\overline{\alpha}) = \sum_{n=k}^{\infty} s(n,k;\overline{\alpha})\frac{t^{n-1}}{(n-1)!},$$

and from (2.2) we get

$$\begin{aligned} \phi_k'(t;\overline{\alpha}) &= \sum_{n=k}^{\infty} s(n-1,k-1;\overline{\alpha})\frac{t^{n-1}}{(n-1)!} + \alpha_k \sum_{n=k+1}^{\infty} s(n-1,k;\overline{\alpha})\frac{t^{n-1}}{(n-1)!} \\ &\qquad - t\sum_{n=k+1}^{\infty} s(n-1,k;\overline{\alpha})\frac{t^{n-2}}{(n-2)!} \\ &= \phi_{k-1}(t;\overline{\alpha}) + \alpha_k\phi_k(t,\overline{\alpha}) - t\phi_k'(t;\overline{\alpha}); \end{aligned}$$

hence,

$$\phi_k'(t;\overline{\alpha}) - \frac{\alpha_k}{(1+t)}\phi_k(t;\overline{\alpha}) = \frac{1}{1+t}\phi_{k-1}(t;\overline{\alpha}).$$

Solving this difference-differential equation, we obtain (2.3).

Theorem 2.3: The numbers $s(n,k;\overline{\alpha})$ have the following explicit form:

$$s(n,k;\overline{\alpha}) = \sum_{r=k}^{n}\frac{n!}{r!}(-1)^{n-r}\sum\frac{1}{\ell_1\ell_2\ldots\ell_r}\sum_{j=0}^{k}\frac{\alpha_j^r}{(\alpha_j)_k}, \tag{2.5}$$

where, in the second sum, the summation extends over all ordered n-tuples of integers $(\ell_1,\ell_2,\ldots,\ell_r)$ satisfying the conditions $\ell_1+\ell_2+\cdots+\ell_r = n$ and $\ell_i \ge 1$, $i=1,2,\ldots,r$.

Proof: From (2.3),

$$\begin{aligned} \phi_k(t;\overline{\alpha}) &= \sum_{j=0}^{k}\frac{(1+t)^{\alpha_j}}{(\alpha_j)_k} = \sum_{j=0}^{k}\frac{e^{\alpha_j\log(1+t)}}{(\alpha_j)_k} \\ &= \sum_{j=0}^{k}\frac{1}{(\alpha_j)_k}\sum_{r=0}^{\infty}\frac{(\alpha_j\log(1+t))^r}{r!} = \sum_{j=0}^{k}\frac{1}{(\alpha_j)_k}\sum_{r=0}^{\infty}\frac{\alpha_j^r}{r!}\left(\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{t^\ell}{\ell}\right)^r, \end{aligned}$$

and using Cauchy's rule of multiplication of infinite series, we get (2.5).

In the following, we find a relationship between $s(n, k)$ and $s(n, k; \overline{\alpha})$. From (2.1), we have

$$(t)_n = \sum_{k=0}^{n} s(n, k; \overline{\alpha})\,(t/\alpha)_k;$$

hence,

$$\begin{aligned}\sum_{i=0}^{n} s(n, i)t^i &= \sum_{k=0}^{n} s(n, k; \overline{\alpha}) \sum_{i=0}^{k} s_\alpha(k, i)t^i \\ &= \sum_{i=0}^{n}\left(\sum_{k=i}^{n} s(n, k; \overline{\alpha}) s_\alpha(k, i)\right)t^i.\end{aligned}$$

Equating the coefficients of t^i on both sides, we get

$$s(n, i) = \sum_{k=i}^{n} s(n, k; \overline{\alpha}) s_\alpha(k, i). \tag{2.6}$$

Similarly, we can express $s(n, k; \overline{\alpha})$ in terms of $s(n, k)$. Since

$$(t)_n = \sum_{k=0}^{n} s(n, k)t^k,$$

we have, from (1.4) and (2.1),

$$(t)_n = \sum_{k=0}^{n} s(n, k) \sum_{i=0}^{k} S_\alpha(k, i)\,(t/\alpha)_i;$$

therefore,

$$\sum_{i=0}^{n} s(n, i; \overline{\alpha})\,(t/\alpha)_i = \sum_{i=0}^{n}\left(\sum_{k=i}^{n} s(n, k) S_\alpha(k, i)\right)(t/\alpha)_i,$$

and hence,

$$s(n, i; \overline{\alpha}) = \sum_{k=i}^{n} s(n, k) S_\alpha(k, i). \tag{2.7}$$

Also we can express $S_\alpha(n, k)$ in terms of $s(n, k; \overline{\alpha})$. Since

$$t^n = \sum_{k=0}^{n} S(n, k)\,(t)_k = \sum_{k=0}^{n} S(n, k) \sum_{i=0}^{k} s(k, i; \overline{\alpha})\,(t/\alpha)_i,$$

we have

$$\sum_{i=0}^{n} S_\alpha(n, i)\,(t/\alpha)_i = \sum_{i=0}^{n}\left(\sum_{k=i}^{n} S(n, k) s(k, i; \overline{\alpha})\right)(t/\alpha)_i;$$

hence,

$$S_\alpha(n, i) = \sum_{k=i}^{n} S(n, k) s(k, i; \overline{\alpha}). \tag{2.8}$$

Combining equations (2.6) and (2.7), we get an orthogonality relation of $s_\alpha(n, k)$ and $S_\alpha(n, k)$. Since

$$s(n,i)=\sum_{k=i}^{n}s_\alpha(k,i)\sum_{\ell=k}^{n}s(n,i)S_\alpha(\ell,k)=\sum_{\ell=i}^{n}\left(\sum_{k=i}^{\ell}s_\alpha(k,i)S_\alpha(\ell,k)\right)s(n,\ell);$$

hence,

$$\sum_{k=i}^{\ell}S_\alpha(\ell,k)s_\alpha(k,i)=\delta_{\ell i},$$

where $\delta_{\ell i}$ is Kronecker's delta.

3. THE MULTIPARAMETER NONCENTRAL STIRLING NUMBERS OF THE SECOND KIND

Definition: Let t be a real number, n a nonnegative integer, and $\overline{\alpha}=(\alpha_0,\alpha_1,\dots,\alpha_{n-1})$, where $\alpha_0<\alpha_1<\cdots<\alpha_{n-1}$ are real numbers.

We define $S(n,k;\alpha_0,\alpha_1,\dots,\alpha_{n-1})$, briefly denoted by $S(n,k;\overline{\alpha})$, the multiparameter noncentral Stirling numbers of the second kind with parameters $\overline{\alpha}=(\alpha_0,\alpha_1,\dots,\alpha_{n-1})$, by

$$(t/\alpha)_n=\sum_{k=0}^{n}S(n,k;\overline{\alpha})\,(t)_k, \tag{3.1}$$

where $S(0,0;\overline{\alpha})=1$ and $S(n,k;\overline{\alpha})=0$ for $k>n$.

Theorem 3.1: The numbers $S(n,k;\overline{\alpha})$ satisfy the recurrence relation

$$S(n,k;\overline{\alpha})=S(n-1,k-1;\overline{\alpha})+(k-\alpha_{n-1})S(n-1,k;\overline{\alpha}). \tag{3.2}$$

Proof: Since $(t/\alpha)_n=(t/\alpha)_{n-1}(t-\alpha_{n-1})=(t/\alpha)_{n-1}[(t-k)+(k-\alpha_{n-1})]$, we obtain, from (3.1),

$$\sum_{k=0}^{n}S(n,k;\overline{\alpha})\,(t)_k=(t-k)\sum_{k=0}^{n-1}S(n-1,k;\overline{\alpha})\,(t)_k+(k-\alpha_{n-1})\sum_{k=0}^{n-1}S(n-1,k;\overline{\alpha})\,(t)_k,$$

which gives us (3.2).

We discuss the following special cases:

i) If $\alpha_i=\alpha$, $i=0,1,\dots,n-1$, then from (3.2) we have

$$S(n,k;\alpha)=S(n-1,k-1;\alpha)+(k-\alpha)S(n-1,k;\alpha),$$

where $S(n,k;\alpha)$ denotes the noncentral Stirling numbers of the second kind as defined by Koutras [6].

ii) If $\alpha_i=0$, $i=0,1,\dots,n-1$, then from (3.2) we have

$$S(n,k)=S(n-1,k-1)+kS(n-1,k),$$

where $S(n,k)$ denotes the Stirling numbers of the second kind (see [9]).

iii) If $\alpha_i=i$, $i=0,1,\dots,n-1$, then $S(n,k;\overline{\alpha})$ reduces to the C-numbers, where $r=1$, i.e., $C(n,k;1)$ (see [2]).

In the following, we find a relationship between $s_\alpha(n,k)$ and $S(n,k;\overline{\alpha})$.

From (3.1) we have

$$(t/\alpha)_n = \sum_{k=0}^{n} S(n,k;\overline{\alpha})\,(t)_k = \sum_{k=0}^{n} S(n,k;\overline{\alpha}) \sum_{i=0}^{k} s(k,i)t^i;$$

hence,

$$\sum_{i=0}^{n} s_\alpha(n,i)t^i = \sum_{i=0}^{n}\left(\sum_{k=i}^{n} S(n,k;\overline{\alpha})\, s(k,i)\right)t^i,$$

and equating the coefficients of t^i on both sides, we get

$$s_\alpha(n,i) = \sum_{k=i}^{n} S(n,k;\overline{\alpha})\, s(k,i). \tag{3.3}$$

Similarly, we have

$$(t/\alpha)_n = \sum_{k=0}^{n} s_\alpha(n,k)t^k = \sum_{k=0}^{n} s_\alpha(n,k) \sum_{i=0}^{k} S(k,i)\,(t)_i;$$

therefore,

$$\sum_{i=0}^{n} S(n,i;\overline{\alpha})\,(t)_i = \sum_{i=0}^{n}\left(\sum_{k=i}^{n} s_\alpha(n,k)S(k,i)\right)(t)_i,$$

and hence,

$$S(n,i;\overline{\alpha}) = \sum_{k=i}^{n} s_\alpha(n,k)S(k,i). \tag{3.4}$$

Also, we can express $S(n,k)$ in terms of $S(n,k;\overline{\alpha})$. It follows from (1.4) that

$$t^n = \sum_{k=0}^{n} S_\alpha(n,k)\,(t/\alpha)_k = \sum_{k=0}^{n} S_\alpha(n,k) \sum_{i=0}^{n} S(k,i;\overline{\alpha})\,(t)_i.$$

Thus,

$$\sum_{i=0}^{n} S(n,i)\,(t)_i = \sum_{i=0}^{n}\left(\sum_{k=i}^{n} S_\alpha(n,k)S(k,i,\overline{\alpha})\right)(t)_i,$$

implying that

$$S(n,i) = \sum_{k=i}^{n} S_\alpha(n,k)S(k,i;\overline{\alpha}). \tag{3.5}$$

Moreover, we can find a relationship between $s(n,k;\overline{\alpha})$, $s_\alpha(n,k)$, and $s(n,k;\alpha)$, as follows. From (2.6) and (1.9) in [6], we get

$$\sum_{\ell=i}^{n}\binom{n}{\ell}(-\alpha)_{n-\ell}\, s(\ell,i;\alpha) = \sum_{\ell=i}^{n} s(n,\ell;\overline{\alpha})s_\alpha(\ell,i);$$

hence,

$$\sum_{\ell=i}^{n}\left(s(n,\ell;\overline{\alpha})s_\alpha(\ell,i) - \binom{n}{\ell}(-\alpha)_{n-\ell}\, s(n,i;\alpha)\right) = 0. \tag{3.6}$$

Similarly, from (2.5) and equation (2.5a) in [6], we get

$$\sum_{k=i}^{n}\left(S_\alpha(n,k)S(k,i;\overline{\alpha}) - \binom{n}{k}\alpha^{n-k} S(k,i;\alpha)\right) = 0. \tag{3.7}$$

4. APPLICATIONS

i. From (2.6), and since

$$s(n,i)=\sum_{k=i}^{n}(-1)^{i}L(n,k)s(k,i),$$

where $L(n,k)$ denotes the Lah numbers (see [3]); hence, we obtain the combinatorial identity

$$\sum_{k=i}^{n}\left((-1)^{i}L(n,k)s(k,i)-s(n,k;\overline{\alpha})s_{\alpha}(k,i)\right)=0. \tag{4.1}$$

Similarly, from (2.6), and since

$$s(n,i)=r^{-i}\sum_{k=i}^{n}C(n,k,r)s(k,i),$$

where $C(n,k,r)$ denotes the C-numbers (see [3]), we have the combinatorial identity

$$\sum_{k=i}^{n}\left(s(n,k;\alpha)s_{\alpha}(k,i)-r^{-i}C(n,k,r)s(k,i)\right)=0. \tag{4.2}$$

ii. We find an orthogonality relation of $s(n,k;\overline{\alpha})$ and $S(n,k;\overline{\alpha})$. From (2.1) and (3.1), we get

$$\begin{aligned}(t)_n&=\sum_{k=0}^{n}s(n,k;\overline{\alpha})\,(t/\alpha)_k=\sum_{k=0}^{n}s(n,k;\overline{\alpha})\left(\sum_{i=0}^{k}S(k,i;\overline{\alpha})\,(t)_i\right)\\&=\sum_{i=0}^{n}\left(\sum_{k=i}^{n}s(n,k;\overline{\alpha})S(k,i;\overline{\alpha})\right)(t)_i;\end{aligned}$$

hence,

$$\sum_{k=i}^{n}s(n,k;\overline{\alpha})S(k,i;\overline{\alpha})=\delta_{ni}, \tag{4.3}$$

where δ_{ni} is Kronecker's delta.

iii. Let $M_{j,k}(x)$ denote the B-spline of Curry Schoenberg with knots $\xi_j<\xi_{j+1}<\cdots<\xi_{j+k}$ $(j\in Z, k=1,2,\ldots)$ as defined in [7]. The moments $\mu_{\ell}(k,\xi)$ of the B-spline $M_{j,k}(x)$ when the index j is equal to 0 is given by

$$\mu_{\ell}(k,\xi)=\int_{-\infty}^{\infty}x^{\ell}M_{0,k}(x)\,dx,\ \ \ell=0,1,\ldots;\ k=1,2,\ldots;\quad \xi=(\xi_0,\xi_1,\ldots,\xi_k).$$

From (3.3) and Proposition 3.1 in [7], we get

$$\mu_{n-i}(k,\xi)=\binom{n}{k}^{-1}\sum_{k=i}^{n}S(n,k;\overline{\alpha})s(k,i). \tag{4.4}$$

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