ON THE INTEGRITY OF CERTAIN FIBONACCI SUMS

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1. AIM OF THE PAPER

Some years ago we were rather surprised at the integrity of the infinite sum

$$\sum_{i=0}^{\infty} F_i / 2^i = 2 \quad (F_i \text{ the } i^{\text{th}} \text{ Fibonacci number}) \tag{1.1}$$

which was obtained in [2] as a by-product result. Our mathematical curiosity led us to investigate (see [1] and [3]) the rational values (in particular, the integral values) of r for which the sum

$$\sum_{i=0}^{\infty} F_i / r^i \tag{1.2}$$

gives a positive integer.

The aim of this paper is to extend the results established in [1] and [3] by finding the sef of *all rational* values of r for which the sum

$$S(r,n) = \sum_{i=0}^{\infty} F_{ni} / r^i \quad (r \neq 0) \tag{1.3}$$

[n is an arbitrary natural number, r is an arbitrary (nonzero) real quantity) gives a *positive* integer k. Since both r and k turn out to be Fibonacci number ratios, the results established in this paper can be viewed as a particular kind of Fibonacci identities that are believed to be new [see (4.7) and (4.8)].

Throughout the paper we shall make use of the following properties of the Fibonacci numbers and of the Lucas numbers L_n which are either available in [5] and [11] or can be readily derived by using the Binet forms for F_n and L_n:

$$F_{2n} = F_n L_n, \tag{1.4}$$

$$5F_n^2 = L_n^2 - 4(-1)^n, \tag{1.5}$$

$$L_{2n} - 2(-1)^n = 5F_n^2, \tag{1.6}$$

$$F_n \text{ divides } F_k \text{ iff } n \text{ divides } k \text{ (for } n \geq 3), \tag{1.7}$$

$$L_n \equiv L_k \pmod 5 \text{ iff } n \equiv k \pmod 4, \tag{1.8}$$

$$L_{n+k} - (-1)^k L_{n-k} = 5F_n F_k, \tag{1.9}$$

$$L_{n+k} + (-1)^k L_{n-k} = L_n L_k. \tag{1.10}$$

2. THE VALUES OF r FOR WHICH $S(r, n)$ IS A POSITIVE INTEGER

The closed-form expression

$$S(r,n) = \frac{rF_n}{r^2 - rL_n + (-1)^n} \tag{2.1}$$

which is valid if and only if the inequality

$$|r| > \alpha^n = [(1+\sqrt{5})/2]^n \tag{2.2}$$

is satisfied, can be obtained as a particular case of formula (5.2) in [6]. On the other hand, (2.1) and (2.2) can be obtained with the aid of the Binet form and the geometric series formula. If (2.2) is not satisfied, then $S(r,n)$ diverges. Now let us ask ourselves the following question:

"For which values r_k of r does $S(r,n)$ equal a positive integer k?"

To answer this question, let us equate the right-hand side of (2.1) to k, thus obtaining the second-degree equation

$$kr^2 - (F_n + kL_n)r + k(-1)^n = 0 \tag{2.3}$$

in the unknown r, the roots of which are

$$r_1 = \frac{F_n + kL_n + \sqrt{D}}{2k}, \qquad r_2 = \frac{F_n + kL_n - \sqrt{D}}{2k}, \tag{2.4}$$

where

$$D = (F_n + kL_n)^2 - 4k^2(-1)^n. \tag{2.5}$$

Observe that, by (1.4) and (1.5), D can be equivalently expressed as

$$D = (5k^2+1)F_n^2 + 2kF_{2n}. \tag{2.6}$$

After some tedious manipulations involving the use of the Binet forms, it is seen that, for $k, n \geq 1$,

$$\begin{cases} r_1 > \alpha^n, \\ r_1 = (-1)^n / r_2. \end{cases} \tag{2.7}$$

From (2.7), we get the inequality $|r_2| < \alpha^n$, so that only the "plus" sign must be considered in (2.4) [see (2.2)]. It follows that $S(r,n)$ equals a positive integer k iff

$$r \equiv r_1 \stackrel{\text{def}}{=} r(n) = \frac{F_n + kL_n + \sqrt{D}}{2k}. \tag{2.8}$$

3. THE RATIONAL VALUES OF r FOR WHICH $S(r, n)$ IS A POSITIVE INTEGER

Since the numbers $r(n)$ defined by (2.8) are, in general, irrational, let us ask ourselves whether or not there exist rational values of them. This is equivalent to asking whether there exist positive integers k for which D is the square of an integer: the answer is in the affirmative, as we shall see in the sequel.

In [1] it has been proved that the set of rational numbers r for which $S(r,1)$ is a positive integer is

$$\{F_{2h+1}/F_{2h} | h = 1, 2, \dots\}; \tag{3.1}$$

moreover,

$$S(F_{2h+1}/F_{2h}, 1) = F_{2h}F_{2h+1}. \tag{3.2}$$

For the general case (i.e., $n \ge 1$), we state the following

Theorem 1 (Main Result): Let $S(r,n) = \sum_{i=0}^{\infty} F_{ni}/r^i$.

(i) If n is even, then the set of all rational numbers r for which $S(r,n)$ is a positive integer is

$$\{F_{(h+1)n}/F_{hn} | h = 1, 2, \dots\}; \tag{3.3}$$

moreover,

$$S(F_{(h+1)n}/F_{hn}, n) = F_{(h+1)n}F_{hn}/F_n. \tag{3.4}$$

(ii) If n is odd, then the set of all rational numbers r for which $S(r,n)$ is a positive integer is

$$\{F_{(2h+1)n}/F_{2hn} | h = 1, 2, \dots\}; \tag{3.5}$$

moreover,

$$S(F_{(2h+1)n}/F_{2hn}, n) = F_{(2h+1)n}F_{2hn}/F_n. \tag{3.6}$$

By means of formula (11) in [4], it can be proved that

$$\frac{F_{(h+1)n}F_{hn}}{F_n} = \sum_{i=1}^{h} F_{2ni} \quad (n \text{ even}) \tag{3.7}$$

and

$$\frac{F_{(2h+1)n}F_{2hn}}{F_n} = \sum_{i=1}^{h}\sum_{j=1}^{n} F_{4ni-2(n-j)-1} \quad (n \text{ odd}). \tag{3.7'}$$

Since (3.7) and (3.7') are nothing but marginal results, their detailed proofs are omitted.

To prove Theorem 1 we have to prove the following two theorems.

Theorem 2:

(i) If n is even, the discriminant $D = (5k^2+1)F_n^2 + 2kF_{2n}$ [see (2.6)] is the square of an integer iff

$$k = F_{(h+1)n}F_{hn}/F_n. \tag{3.8}$$

(ii) If n is odd, the discriminant D is the square of an integer iff

$$k = F_{(2h+1)n}F_{2hn}/F_n. \tag{3.9}$$

Theorem 3:

(i) If n is even and (3.8) holds, then [cf. (2.8)] $r(n) = F_{(h+1)n}/F_{hn}$.

(ii) If n is odd and (3.9) holds, then $r(n) = F_{(2h+1)n}/F_{2hn}$.

Proof of Theorem 2: We shall prove that, if D is the square of a generic integer, then k must necessarily be either of the form (3.8) (if n is even) or of the form (3.9) (if n is odd). Let us suppose that $D = X^2$ $(X \in \mathbb{N})$. From (2.6) we can write

$$5k^2F_n^2 + 2kF_{2n} + F_n^2 - X^2 = 0, \tag{3.10}$$

whence we have

$$k = [-F_{2n} \pm \sqrt{F_{2n}^2 - 5F_n^2(F_n^2 - X^2)}]/(5F_n^2). \tag{3.11}$$

After some simple manipulations involving the use of (1.4), and taking into account that k must be positive (by hypothesis), (3.11) can be rewritten as

$$k = [-L_n + \sqrt{L_n^2 - 5F_n^2 + 5X^2}]/(5F_n). \tag{3.12}$$

Now let us distinguish two cases according to the parity of n.

Case 1: n is even.

From (1.5), (3.12) becomes

$$k = [-L_n + \sqrt{5X^2 + 4}]/(5F_n). \tag{3.13}$$

For k to be an integer, at least we must have that

$$5X^2 + 4 = Q^2 \quad (Q \in \mathbb{N}). \tag{3.14}$$

The solution in integers of the above *Pell equation* is (e.g., see Lemma 1 in [7] or formulas (3.7)-(3.8) in [1])

$$Q = L_{2s}, \; X = F_{2s} \quad (s = 0, 1, 2, \ldots), \tag{3.15}$$

so that, from (3.13)-(3.15), we have

$$k = (L_{2s} - L_n)/(5F_n). \tag{3.16}$$

Now, for k to be a positive integer, both the inequality

$$2s > n, \tag{3.17}$$

and the congruences

$$L_{2s} - L_n \equiv 0 \pmod 5, \tag{3.18}$$

$$L_{2s} - L_n \equiv 0 \pmod{F_n} \tag{3.19}$$

must simultaneously hold. Let us find conditions on s for (3.18) and (3.19) to be satisfied. From (3.18) and (1.8), we see that the congruence

$$2s \equiv n \pmod 4 \tag{3.20}$$

must hold. Now let us rewrite the numerator of (3.16) as

$$L_{2s} - L_n = L_{(2s+n)/2+(2s-n)/2} - L_{(2s+n)/2-(2s-n)/2} \tag{3.21}$$

and observe that, in virtue of (3.20), the integer $(2s-n)/2$ must be even. Under this condition, we can use (1.9) to obtain

$$L_{2s} - L_n = 5F_{(2s-n)/2}F_{(2s+n)/2}. \tag{3.22}$$

First, let us consider the case $n=2$. From (3.16) and (3.22), we obtain the equality $k = F_{s-1}F_{s+1}$, where, from (3.17) and (3.20), s ranges over all odd integers greater than 1. It follows that the above equality can be rewritten as $k = F_{2h}F_{2(h+1)}$ $(h = 1, 2, \dots)$ [cf. (3.8) for $n=2$ and take into account that $F_2 = 1$].

For $n \geq 4$, the equality (3.22) shows clearly that (3.19) is satisfied iff [see (1.7)]

$$\frac{2s-n}{2} \left(\text{or } \frac{2s+n}{2} \right) \equiv 0 \pmod{n}.$$

Taking (3.17) into account, the above congruence can be written as

$$\frac{2s-n}{2} = hn \quad (h = 1, 2, \dots). \tag{3.23}$$

From (3.23) we have

$$\frac{2s+n}{2} = (h+1)n \quad (h = 1, 2, \dots). \tag{3.24}$$

Finally, from (3.16) and (3.22)-(3.24), we obtain the desired result

$$k = \frac{5F_{hn}F_{(h+1)n}}{5F_n} = \frac{F_{hn}F_{(h+1)n}}{F_n} \quad (h = 1, 2, \dots).$$

Case 2: n is odd.

The proof is analogous to that of Case 1, so it is simply sketched. From (1.5), the equality (3.12) and the Pell equation (3.14) become

$$k = [-L_n + \sqrt{5X^2 - 4}]/(5F_n) \tag{3.13′}$$

and

$$5X^2 - 4 = Q^2 \quad (Q \in \mathbb{N}), \tag{3.14′}$$

respectively. The solution in integers of (3.14′) is (see Lemma 2 in [7])

$$Q = L_{2s+1}, \quad X = F_{2s+1} \quad (s = 0, 1, 2, \dots). \tag{3.15′}$$

Therefore, by means of the same argument as that of Case 1, we get the following relations:

$$k = (L_{2s+1} - L_n)/(5F_n), \tag{3.16′}$$

$$2s+1 > n, \tag{3.17′}$$

$$2s+1 \equiv n \pmod{4}, \tag{3.20′}$$

$$L_{2s+1} - L_n = 5F_{(2s+1-n)/2}F_{(2s+1+n)/2}. \tag{3.22′}$$

Taking (3.17′) into account, and recalling that n is odd and $(2s+1-n)/2$ must be even [in virtue of (3.20′)], we can write [see (1.7)]

$$\frac{2s+1-n}{2} = 2hn \quad (h = 1, 2, \dots). \tag{3.23′}$$

From (3.23′) we have

$$\frac{2s+1+n}{2} = (2h+1)n \quad (h = 1, 2, \ldots). \tag{3.24′}$$

Finally, from (3.16′) and (3.22′)-(3.24′), we obtain

$$k = \frac{F_{2hn}F_{(2h+1)n}}{F_n} \quad (h = 1, 2, \ldots) \quad \text{Q.E.D.}$$

Proof of Theorem 3: Let us distinguish two cases according to the parity of n.

Case 1: n is even.

First, let us replace k by the right-hand side of (3.8) in (2.6), thus obtaining

$$D = 5F_{(h+1)n}^2 F_{hn}^2 + F_n^2 + 2F_{(h+1)n}F_{hn}L_n \stackrel{\text{def}}{=} D(n), \tag{3.25}$$

where (1.4) has been invoked. With the aid of (1.9) and (1.5), the relation (3.25) can be rewritten as

$$\begin{aligned} D(n) &= 5[L_{(2h+1)n} - L_n / 5]^2 + F_n^2 + 2L_n(L_{(2h+1)n} - L_n)/5 \\ &= (L_{(2h+1)n}^2 - L_n^2)/5 + F_n^2 = (L_{(2h+1)n}^2 - L_n^2)/5 + (L_n^2 - 4)/5 \\ &= (L_{(2h+1)n}^2 - 4)/5 = F_{(2h+1)n}^2. \end{aligned} \tag{3.26}$$

Then, let us replace k by the right-hand side of (3.8) and D by $D(n)$ in (2.8), thus obtaining

$$r(n) = \frac{F_n^2 + L_nF_{(h+1)n}F_{hn} + F_nF_{(2h+1)n}}{2F_{(h+1)n}F_{hn}} \stackrel{\text{def}}{=} \frac{N_1}{N_2}. \tag{3.27}$$

Now, it is plain that, in order to prove the theorem, it is sufficient [cf. (3.3)] to prove that $N_1 = 2F_{(h+1)n}^2$. In fact, using (1.9), we get, from (3.27), the equality

$$N_1 = F_n^2 + L_n(L_{(2h+1)n} - L_n)/5 + (L_{2(h+1)n} - L_{2hn})/5,$$

whence, using (1.5) and (1.10), we have

$$\begin{aligned} 5N_1 &= -4 + L_nL_{(2h+1)n} + L_{2(h+1)n} - L_{2hn} \\ &= -4 + L_{2(h+1)n} + L_{2hn} + L_{2(h+1)n} - L_{2hn} = 2(L_{2(h+1)n} - 2). \end{aligned} \tag{3.28}$$

Finally, using (1.6), equality (3.28) becomes $5N_1 = 10F_{(h+1)n}^2$, whence, as desired, we obtain $N_1 = 2F_{(h+1)n}^2$.

Case 2: n odd.

The proof is obtained by replacing k by the right-hand side of (3.9) and by using the same properties of Fibonacci numbers as those used in Case 1. Thus, the proof is omitted for the sake of brevity. We confine ourselves to putting into evidence that, in this case, we have

$$D(n) = F_{(4h+1)n}^2 \tag{3.26′}$$

and

$$N_1 = F_n^2 + L_n F_{(2h+1)n} F_{2hn} + F_n F_{(4h+1)n} = 2F_{(2h+1)n}^2. \quad \text{Q.E.D.}$$

4. CONCLUDING REMARKS

The Fibonacci-type sum $S(r, n)$ has been investigated and the rational values of r for which this sum is a positive integer have been determined. We can observe that, as required [see (2.2)],

$$\frac{F_{(h+1)n}}{F_{hn}}, \; \frac{F_{(2h+1)n}}{F_{2hn}} > \alpha^n. \tag{4.1}$$

More particularly, with the aid of the Binet form, we can see that the two quantities on the left-hand side of (4.1) tend to α^n as h tends to infinity.

Remark 1: Let us answer the question of whether or not there exist *integral* values of r for which $S(r, n)$ is a positive integer. From (1.7), (3.3), and (3.5), and taking into account that $F_2 = 1$ divides F_k for all k, it follows that the only integral values of r for which $S(r, n)$ is a positive integer are

$$r = F_{2n} / F_n = L_n \quad (n = 2, 4, \ldots) \tag{4.2}$$

and

$$r = F_3 / F_2 = 2 \quad [\text{cf. } (1.1)]. \tag{4.3}$$

Recalling that $L_0 = 2$, it is apparent that the set of such values of r is constituted by all the even-subscripted Lucas numbers.

Remark 2: The *generalized Fibonacci numbers* $U_i(m)$ have been considered in [1], [3], [6], [9], and [10]. These numbers are defined by

$$U_0(m) = 0, \; U_1(m) = 1, \; U_i(m) = mU_{i-1}(m) + U_{i-2}(m) \; \text{ if } i > 1, \tag{4.4}$$

where m is an arbitrary natural number. They give the Fibonacci numbers and the Pell numbers when $m = 1$ and 2, respectively. Once F has been replaced by U in (1.3), the solution in integers of the Pell equations (e.g., see [8], pp. 305-09)

$$(m^2 + 4)X^2 \pm 4 = Q^2 \tag{4.5}$$

allows to prove that the results established in Theorem 1 apply to the numbers $U_i(m)$ as well, provided the inequality $|r| > [(m + \sqrt{m^2 + 4}) / 2]^n$ is satisfied.

Finally, we point out that the results established in this paper give rise to the following Fibonacci identities which we hope will be of some interest to the reader:

$$\sum_{i=0}^{\infty} \frac{F_{ni} F_{hn}^i}{F_{(h+1)n}^i} = \frac{F_{(h+1)n} F_{hn}}{F_n} \quad (n \geq 2 \text{ even}, \; h \geq 1), \tag{4.6}$$

$$\sum_{i=0}^{\infty} \frac{F_{ni} F_{2hn}^i}{F_{(2h+1)n}^i} = \frac{F_{(2h+1)n} F_{2hn}}{F_n} \quad (n \geq 1 \text{ odd}, \; h \geq 1). \tag{4.7}$$

Observe that the right-hand sides of (4.6) and (4.7) can be replaced by those of (3.7) and (3.7′) according to the parity of n. As particular instances, letting $h=1$ in (4.6) yields

$$\sum_{i=0}^{\infty} \frac{F_{ni}}{L_n^i} = F_{2n} \quad (n \geq 2 \text{ even}), \tag{4.6′}$$

whereas letting $n=h=1$ in (4.7) yields (1.1).

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