ROOTS OF SEQUENCES UNDER CONVOLUTIONS

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1. INTRODUCTION

The usual convolution of the sequences $\{r_n\}$ and $\{s_n\}$ is defined to be the sequence $\{t_n\}$ given by $t_n = \sum_{i=0}^{n} r_i s_{n-i}$ $(n \geq 0)$. The usual convolution comes out naturally from the product of the generating functions of the sequences $\{r_n\}$ and $\{s_n\}$:

$$\left(\sum_{n=0}^{\infty} r_n x^n\right)\left(\sum_{n=0}^{\infty} s_n x^n\right) = \sum_{n=0}^{\infty} t_n x^n .$$

This "usual" convolution is also called the Cauchy product. We define the k^{th} power $\{r_n^{(k)}\}$ of the sequence $\{r_n\}$ under the usual convolution as follows:

$$r_n^{(1)} = r_n; \quad r_n^{(k)} = \sum_{i=0}^{n} r_i r_{n-i}^{(k-1)} \quad (k \geq 2).$$

In other words, $r_n^{(k)} = \sum_{i_1+\cdots+i_k=n} r_{i_1} r_{i_2} \cdots r_{i_k}$.

Using the terminology of [6], the k^{th} power under the usual convolution is the $(k-1)^{\text{th}}$ iterated convolution.

The binomial convolution ([2], §7.6) of the sequences $\{r_n\}$ and $\{s_n\}$ is defined to be the sequence $\{u_n\}$ given by

$$u_n = \sum_{i=0}^{n} \binom{n}{i} r_i s_{n-i} \quad (n \geq 0).$$

This convolution arises from the product of the exponential generating functions. Namely,

$$\left(\sum_{n=0}^{\infty} r_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} s_n \frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} u_n \frac{x^n}{n!}.$$

We define the k^{th} power $\{r_n^{[k]}\}$ under the binomial convolution of $\{r_n\}$ naturally as follows:

$$r_n^{[1]} = r_n; \quad r_n^{[k]} = \sum_{i=0}^{n} \binom{n}{i} r_i r_{n-i}^{[k-1]} \quad (k \geq 2).$$

Thus,

$$r_n^{[k]} = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1}\binom{n-i_1}{i_2}\cdots\binom{n-i_1-\cdots-i_{k-2}}{i_{k-1}} r_{i_1} r_{i_2} \cdots r_{i_k} = \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1!\, i_2! \cdots i_k!} r_{i_1} r_{i_2} \cdots r_{i_k}.$$

In this paper we shall study solutions of the equations $\{r_n^{(k)}\} = \{s_n\}$ and $\{r_n^{[k]}\} = \{s_n\}$ in $\{r_n\}$, where $\{s_n\}$ is a fixed sequence (see Sections 2 and 3). The solutions can be referred to as the k^{th}

roots of $\{s_n\}$ under the usual and the binomial convolution, respectively. In Section 4, roots of sequences under a general weighted convolution are briefly considered.

If $s_n = 0$ for all n, then $r_n = 0$ for all n is the only solution for the equations. Therefore, we may confine ourselves to sequences $\{s_n\}$ such that $s_n \neq 0$ for some n. The least n with $s_n \neq 0$ will be denoted by $\chi(s_n)$.

Since an arithmetic function $f(n)$ is uniquely determined by the corresponding sequence $\{f(1), f(2), f(3), \ldots\}$, it follows that the study of the roots of sequences considered here is similar to the study of roots of arithmetic functions already made in papers [1] and [7] under Dirichlet convolution and in paper [3] under "exponential Narkiewicz" convolution. In [4], roots of arithmetic functions under a generalized Dirichlet convolution are studied.

2. ROOTS OF SEQUENCES UNDER THE USUAL CONVOLUTION

Theorem 1: Let $\{s_n\}$ be a given sequence such that $s_n \neq 0$ for some n. Then the equation $\{r_n^{(k)}\} = \{s_n\}$ has a solution in $\{r_n\}$ if and only if $\chi(s_n)$ is the k^{th} multiple of a nonnegative integer. In this case the equation has exactly k solutions, which can be written as

$$\{r_n\} = \{w_i \rho_n\}, \quad i = 1, \ldots, k, \tag{1}$$

where $\{\rho_n\}$ is one solution and $w_1, \ldots, w_k$ are the k^{th} roots of unity.

Proof: If $\{r_n^{(k)}\} = \{s_n\}$ has a solution, then $k\chi(r_n) = \chi(s_n)$; hence $\chi(s_n)$ is the k^{th} multiple of a nonnegative integer. Conversely, suppose that $\chi(s_n) = km$ for some nonnegative integer m. Then the solutions of $\{r_n^{(k)}\} = \{s_n\}$ can be found as follows. Since $r_n^{(k)} = 0$ for $n < km$, we have $r_n = 0$ for $n < m$. Further, $r_{km}^{(k)} = (r_m)^k$; hence $r_m = (s_{km})^{1/k}$. Finally, the values r_{m+n} $(n \geq 1)$ can be found inductively by using the equations $r_{km+n}^{(k)} = s_{km+n}$ $(n \geq 1)$, whereby it can also be verified that (1) holds. This completes the proof.

For certain sequences $\{s_n\}$, the use of generating functions is a very helpful method of solving the equation $\{r_n^{(k)}\} = \{s_n\}$. Namely, if $r(x)$ and $s(x)$ denote the generating functions of $\{r_n\}$ and $\{s_n\}$, respectively, then $r(x)^k = s(x)$, and hence $r(x) = s(x)^{1/k}$.

We shall illustrate this method in the following examples. For background information on generating functions we refer to [2], [5], and [8].

Example 1: Consider the equation $\{r_n^{(k)}\} = \{a^n\}$, where a is a constant. Then $r(x) = (1-ax)^{-1/k}$ and therefore one solution for the equation is

$$\rho_n = (-1)^n \binom{-1/k}{n} a^n.$$

All solutions can be found by (1). Note that for each integer m, $\rho_n^{(m)} = (-1)^n \binom{-m/k}{n} a^n$. This can be referred to as an $(m/k)^{\text{th}}$ power of the sequence $\{a^n\}$ under the usual convolution.

Example 2: Consider the equation $\{r_n^{(k)}\} = \{s_n\}$, where $\{s_n\}$ is the usual convolution of the sequences $\{a^n\}$ and $\{b^n\}$ with a and b constants. Then $r(x) = (1-ax)^{-1/k}(1-bx)^{-1/k}$ and therefore one solution for the equation is

$$\rho_n = (-1)^n \sum_{i=0}^{n} \binom{-1/k}{i} \binom{-1/k}{n-i} a^i b^{n-i}.$$

All solutions can be found by (1). With $a=(1+\sqrt{5})/2$, $b=(1-\sqrt{5})/2$, this gives the solutions for the equation $\{r_n^{(k)}\}=\{F_{n+1}\}$. Also note that

$$\rho_n^{(m)}=(-1)^n\sum_{i=0}^{n}\binom{-m/k}{i}\binom{-m/k}{n-i}a^ib^{n-i}$$

gives an $(m/k)^{\text{th}}$ power of $\{s_n\}$ under the usual convolution.

Example 3: Let a, b, and c be constants, and $\{\mu_n\}$ the sequence defined by $\mu_0=1, \mu_1=-1$, $\mu_n=0$ $(n\ge 2)$. Then $\{\mu_n\}$ is the inverse of the sequence $\equiv 1$, and the sequence $\{\mu_n c^n\}$ is the inverse of the sequence $\{c^n\}$. Consider the equation $\{r_n^{(k)}\}=\{s_n\}$, where $\{s_n\}$ is the usual convolution of the three sequences $\{a^n\}$, $\{b^n\}$, and $\{\mu_n c^n\}$. Then

$$r(x)=(1-ax)^{-1/k}(1-bx)^{-1/k}(1-cx)^{1/k}.$$

Therefore, one solution is the usual convolution of the three sequences

$$\left\{(-1)^n\binom{-1/k}{n}a^n\right\},\ \left\{(-1)^n\binom{-1/k}{n}b^n\right\},\ \text{and}\ \left\{(-1)^n\binom{1/k}{n}c^n\right\}.$$

That is, one solution is given by

$$\rho_n=(-1)^n\sum_{i_1+i_2+i_3=n}\binom{-1/k}{i_1}\binom{-1/k}{i_2}\binom{1/k}{i_3}a^{i_1}b^{i_2}c^{i_3}.$$

All solutions can be found by (1). With $a=(1+\sqrt{5})/2, b=(1-\sqrt{5})/2, c=1/2$, we obtain the solutions of the equation $\{r_n^{(k)}\}=\{L_n/2\}$. Further, multiplying these solutions by $2^{1/k}$ we obtain the solutions for the equation $\{r_n^{(k)}\}=\{L_n\}$.

Example 4: Since $\chi(F_n)=1$, we see by Theorem 1 that the equation $\{r_n^{(k)}\}=\{F_n\}$ does not have a solution, except for the trivial case $k=1$.

3. ROOTS OF SEQUENCES UNDER THE BINOMIAL CONVOLUTION

Theorem 2: Let $\{s_n\}$ be a given sequence such that $s_n\ne 0$ for some n. Then the equation $\{r_n^{[k]}\}=\{s_n\}$ has a solution in $\{r_n\}$ if and only if $\chi(s_n)$ is the k^{th} multiple of a nonnegative integer. In this case the equation has exactly k solutions, which can be written as

$$\{r_n\}=\{w_i\rho_n\},\quad i=1,\ldots,k, \tag{2}$$

where $\{\rho_n\}$ is one solution and $w_1,\ldots,w_k$ are the k^{th} roots of unity.

Theorem 2 is similar to Theorem 1 in character. Also, Theorem 2 can be proved in a similar way to Theorem 1 and therefore we omit the proof.

The use of exponential generating functions is a helpful method of solving certain equations $\{r_n^{[k]}\}=\{s_n\}$. The following examples will illustrate this method. Here $r_E(x)$ denotes the exponential generating function of $\{r_n\}$.

Example 5: Consider the equation $\{r_n^{[k]}\}=\{\alpha^n\}$. Then $r_E(x)=e^{ax/k}$ and therefore one solution is given by $\rho_n=(a/k)^n$. All solutions can be found by (2).

Example 6: Consider the equation $\{r_n^{[k]}\} = \{(n+1)a^n\}$. Then $r_E(x) = (1+ax)^{1/k}e^{ax/k}$ and therefore one solution is the binomial convolution of the sequences

$$\left\{\frac{1}{k}\left(\frac{1}{k}-1\right)\cdots\left(\frac{1}{k}-(n-1)\right)\right\} \quad \text{and} \quad \left\{\left(\frac{a}{k}\right)^n\right\}.$$

All solutions can be found by (2).

4. A GENERALIZATION

The general weighted convolution of the sequences $\{r_n\}$ and $\{s_n\}$ is defined by

$$\sum_{i=0}^{n} f(n,i)r_i s_{n-i} \quad (n \geq 0),$$

where the weight function $f(n,i)$ is defined for $n \geq 0$ and $0 \leq i \leq n$. If the weight function satisfies the condition

$$f(n,i)f(i,j) = f(n,j)f(n-j,i-j) \tag{3}$$

for all n, i, j with $0 \leq i \leq n$, $0 \leq j \leq i$, then the weighted convolution is associative and we could define powers of sequences under this convolution. We could also consider roots of sequences, and assuming $f(n,i) \neq 0$ for all n and $0 \leq i \leq n$ we could verify that the result of Theorems 1 and 2 also holds with respect to the weighted convolution. We omit the details.

It is easy to see that both the usual and the binomial convolution are special cases of the weighted convolution satisfying (3).

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