THE IRRATIONALITY OF CERTAIN SERIES WHOSE TERMS ARE RECIPROCALS OF LUCAS SEQUENCE TERMS

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1. INTRODUCTION

Let $(P, Q) = 1$ and α and β $(\alpha > \beta)$ be the roots of $x^2 - Px + Q = 0$. The Lucas sequence $U_n = U_n(P, Q)$ and "associated" Lucas sequence $V_n = V_n(P, Q)$ are defined, respectively, by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \ n \geq 0. \tag{0}$$

In 1878 Lucas ([10], p. 225) obtained the following formula:

$$\sum_{n=1}^{\infty} Q^{2^{n-1}r} / U_{2^n r} = \beta^r / U_r, \ r \geq 1.$$

Setting $Q = \pm 1$, it is seen immediately that, if $P^2 - 4Q > 0$, then $\sum_{n=1}^{\infty} 1/U_{2^n r}$ is irrational, since U_r and V_r are integers, $\alpha - \beta$ is irrational, and [from (0)] $\beta^r = (V_r - U_r(\alpha - \beta))/2$ is irrational. Special cases of this result were re-discovered in the mid-1970s for $F_n = U_n(1, -1)$ [6], [7], [9] (see [8] for a number of different methods for summing $\sum_{n=0}^{\infty} 1/F_{2^n}$)

It was now known until recently whether $\sum_{n=1}^{\infty} 1/U_{g(n)}$ is irrational for any values of the parameters P and Q if $g(n) \neq 2^n r$. Then, in 1987, Badea [3] answered a question posed by Erdös and Graham [5] when he proved that $\sum_{n=0}^{\infty} 1/F_{2^n+1}$ is irrational. André-Jeannin [2] has shown that, if $P > 0$ and $Q = \pm 1$, $\sum_{n=1}^{\infty} 1/U_n$ is irrational, and in a recent work [4], Badea proved that $\sum_{n=1}^{\infty} 1/U_{g(n)}$ is irrational for $P > 0$ and $Q < 0$ if $g(n+1) \geq 2g(n) - 1$ for all sufficiently large n.

In this paper we show that, for *all* Lucas sequences with $P > 0$, $(P, Q) = 1$, and $P^2 - 4Q > 0$, $\sum_{n=1}^{\infty} 1/U_{g(n)}$ is irrational if $g(n+1) \geq 2g(n)$ for all sufficiently large n, and show that if $g(n+1) \geq 2g(n) - 1$ for all sufficiently large n and $g(n)$ is even, the result holds for all such positive parameters P and Q. We obtain similar results for $\sum_{n=1}^{\infty} 1/V_{g(n)}$

Let $\sum_{k=1}^{\infty} 1/a_k$ be a series such that $a_{k+1} \geq a_k^2 > 1$ for $k \geq 1$, and denote the partial sum $\sum_{k=1}^{n} 1/a_k$ by x_n / y_n, where x_n and $y_n = a_1 \ldots a_n$ are natural numbers. If, now, $\sum_{k=1}^{\infty} 1/a_k = a/b$, a and b natural numbers, then $a/b = x_n/y_n + \sum_{k=1}^{\infty} 1/a_{n+k}$; that is,

$$0 < ay_n - bx_n = b \cdot \sum_{k=1}^{\infty} \frac{a_1 \ldots a_n}{a_{n+k}}.$$

The sequence $\left\{\frac{a_1 \ldots a_n}{a_{n+k}}\right\}_{n=1}^{\infty}$ is decreasing if $k = 1$ and strictly decreasing if $k > 1$ (implying $\sum_{k=1}^{\infty} \frac{a_1 \ldots a_n}{a_{n+k}}$ is a strictly decreasing function of n), since the ratio of the n^{th} and $(n+1)^{\text{st}}$ term is

$$\frac{a_{n+k+1}}{a_{n+1} \cdot a_{n+k}} \geq \frac{a_{n+k}}{a_{n+1}}$$

which equals 1 if $k=1$ and is >1 if $k>1$. But this implies $\{ay_n - bx_n\}_{n=1}^{\infty}$ is a strictly decreasing sequence of natural numbers, which is impossible; hence, $\sum_{k=1}^{\infty} 1/a_k$ is irrational. We thus have

Theorem A: Let $n \geq 0$. If $\{a_n\}$ is a sequence of integers, except for at most a finite number of terms that are noninteger rationals, and $a_{n+1} \geq a_n^2 > 1$ for all large n, then the series $\sum_{n=0}^{\infty} 1/a_n$ is an irrational number.

This result will suffice to prove Theorems 1, 2, and 4, and all but part (ii) of Theorem 3; for the latter, we require the following stronger criterion due to Badea [2] (rephrased to apply to sequences containing some negative and/or noninteger terms):

Theorem B: Let $n \geq 0$. If $\{a_n\}$ is a sequence of integers, except for at most a finite number of terms that are noninteger rationals, and $a_{n+1} > a_n^2 - a_n + 1 > 0$ for all large n, then the series $\sum_{n=0}^{\infty} 1/a_n$ is an irrational number.

The meanings of U_n and V_n are extended to negative subscripts by defining $U_{-n} = -U_n / Q^n$ and $V_{-n} = V_n / Q^n$. With these definitions, the following known relations hold for all integers m [proofs are readily obtained from (0)].

$$U_{2m} = U_m V_m, \tag{1}$$

$$U_{2m+1} = U_{m+1}^2 - QU_m^2, \tag{2}$$

$$V_{2m} = V_m^2 - 2Q^m, \tag{3}$$

$$V_m > U_m. \tag{4}$$

2. THE THEOREMS

We assume that $Q \neq 0$, $P \geq 1$, and the discriminant $D = P^2 - 2Q > 0$. It is known—and easily shown from (0)—that this assumption assures that $\{U_n\}$ and $\{V_n\}$ are increasing sequences of positive integers.

The proof of the following theorem, for $Q < 0$, is given in [4], but is included here for completeness.

Theorem 1: Let g be an integer-valued function such that $g(n+1) \geq 2g(n) - 1 > 1$ for all large n. The series $\sum_{n=0}^{\infty} 1/U_{g(n)}$ is irrational except possibly when $Q > 0$ and $g(n)$ is odd for infinitely many values of n.

Proof: Let $a_n = U_{g(n)}$ for all $n \geq 0$ and let N be such that $g(n+1) \geq 2g(n) - 1 > 1$ for $n > N$. Assume now that $n > N$.

Case 1. $g(n+1)$ odd. Let $m = m(n)$ be such that $g(n+1) = 2m+1$. Assume Q is negative. By (2),

$$a_{n+1} = U_{g(n+1)} = U_{2m+1} = U_{m+1}^2 - QU_m^2 > U_{m+1}^2.$$

Then $2m+1 = g(n+1)$ implies $m+1 = [g(n+1)+1]/2 \geq g(n)$, so $U_{m+1}^2 \geq U_{g(n)}^2$. Hence,

$$a_{n+1} > U_{g(n)}^2 = a_n^2.$$

Case 2. $g(n+1)$ even. Since $g(n+1) \geq 2g(n)-1$ and $g(n+1)$ is an even integer, $g(n+1) \geq 2g(n)$. Let $g(n+1)=2m$. By (1) and (4),

$$a_{n+1} = U_{g(n+1)} = U_{2m} = U_m V_m > U_m^2.$$

Since $m = g(n+1)/2 \geq g(n)$, we again have $a_{n+1} > a_n^2$. Hence, by Theorem A, $\Sigma_{n=0}^{\infty} 1/U_{g(n)}$ is irrational in each case.

Theorem 2: The series $\Sigma_{n=0}^{\infty} 1/U_{g(n)}$ is irrational if g is an integer-valued function such that $g(n+1) \geq 2g(n) > 1$ for all sufficiently large n.

Proof: Assume that $N>1$ is such that $g(n+1) \geq 2g(n) > 1$ for all $n>N$, and let $n>N$. By Theorem 1, the theorem is true if $g(n+1)$ is even. Let $g(n+1)=2m+1$ and let $a_n = U_{g(n)}$. Since $m=[g(n+1)-1]/2 \geq g(n)-1/2$ is an integer, $m \geq g(n)$. By (1) and (4),

$$a_{n+1} = U_{2m+1} > U_{2m} = U_m V_m > U_m^2 \geq U_{g(n)}^2 = a_n^2,$$

proving the theorem.

We now prove similar theorems for the series $\Sigma_{n=0}^{\infty} 1/V_{g(n)}$. In 1987 Badea [3] proved that $\Sigma_{n=0}^{\infty} 1/L_{2^n}$ is irrational (using Theorem B), and, more generally (in [4]), that $\Sigma_{n=0}^{\infty} 1/V_{g(n)}$ is irrational if $Q=-1$ and $g(n+1) \geq 2g(n)$. André-Jeannin [1] gave a direct proof that, for all positive integers k, $\Sigma_{n=0}^{\infty} (\pm 1)^n / V_{k2^n}$ is irrational, and (in [2]) proved that $\Sigma_{n=0}^{\infty} 1/V_n$ is irrational. Our Theorem 3 includes Badea's results and, for $P>|Q+1|$, André-Jennin's result that $\Sigma_{n=0}^{\infty} 1/V_{k2^n}$ is irrational.

Lemma 1: Let k be a positive integer. If $P>|Q+1|$, then, for all sufficiently large integers m, $kQ^m < V_m - 1$.

Proof: It is easily seen that $|\beta| = |(P-\sqrt{D})/2| < 1$ if and only if $P>|Q+1|$, and that $\alpha > 1$ for all P and Q. Hence, there exists an integer M such that, if $m>M$, then $|\beta|^m < 1/2k$ and $\alpha^m > 4$. It follows that

$$kQ^m = k\alpha^m \beta^m \leq k\alpha^m |\beta|^m < \alpha^m/2 < \alpha^m + \beta^m - 1 = V_m - 1.$$

It is readily shown that $\lim_{n\to\infty} V_{2n+1}/V_n^2 = \alpha > 1$, and this result is sufficient to prove part (i) of Theorem 3. However, it is of interest that $V_{2n+1} > V_n^2$ for *all n*, with one exception.

Lemma 2: If $n>0$, then $V_{2n+1} \geq V_n^2$, with equality holding only when $(P,Q,n)=(3,2,1)$.

Proof: Let $r=\beta/\alpha$ and let

$$f(x) = \frac{\alpha^{2x+1}+\beta^{2x+1}}{(\alpha^x+\beta^x)^2} - 1 = \frac{\alpha(1+r^{2x+1})}{(1+r^x)^2} - 1, \ x \text{ real}.$$

We first observe that $f(1) \geq 0$. Now, since $P^2-4Q>0$,

$$f(1) = V_3/V_1^2 - 1 = (P^2-3Q)/P - 1 > P/4 - 1,$$

so $f(1)>0$ if $P>4$, or if $Q<0$. Since $P^2-4Q>0$ implies $Q<0$ for $P=1$ or 2, $f(1)\le 0$ only if $P=3$ or 4 and $Q>0$. The reader may readily determine that, if $P=3$ or 4, $f(1)\ge 0$ with equality holding only when $P=3$ and $Q=2$.

Case 1. $\beta>0$. Then $0<r<1$. Now,

$$f'(x)=\frac{2\alpha r^x(r^{x+1}-1)\ln r}{(1+r^x)^3}>0,$$

implying that f is a strictly increasing function of x; since $f(n)=V_{2n=1}/V_n^2-1$ and $f(1)\ge 0$, we conclude that $V_{2n=1}>V_n^2$.

Case 2. $\beta<0$. If n is odd, by (3),

$$V_n^2=V_{2n}+2Q^n=V_{2n}+2(\alpha\beta)^n<V_{2n};$$

hence, $V_{2n+1}-V_n^2>V_{2n+1}-V_{2n}>0$. Assume now that n is even. We let $t=-\beta/\alpha$ (so $0<t<1$), define

$$g(x)=\frac{\alpha(1-t^{2x+1})}{(1+t^x)^2}-1,$$

find that g is a strictly increasing function of x, and conclude, since $g(n)=f(n)$ with $t=-r$, that $V_{2n+1}>V_n^2$ in this case, as well.

Theorem 3: Let g be an integer-valued function such that $g(n+1)\ge 2g(n)>1$ for all large n. Then $\sum_{n=0}^{\infty}1/V_{g(n)}$ is irrational

(i) if $g(n)$ is an odd integer for all large n, or
(ii) if $P>|Q+1|$.

Proof: Let $a_n=V_{g(n)}$ for all $n\ge 0$ and let $N>1$ be such that $g(n+1)\ge 2g(n)>1$ for $n>N$. Assume now that $n>N$.

(i) Assume that $g(n+1)$ is odd and let $g(n+1)=2m+1$; since $m=[g(n+1)-1]/2\ge g(n)-1/2$ is an integer, $m\ge g(n)$. Then, by Lemma 2,

$$a_{n+1}=V_{g(n+1)}=V_{2m+1}>V_m^2\ge V_{g(n)}=a_n^2,$$

proving (i).

(ii) Assume that $P>|Q+1|$. We make the additional assumption that, if $r\ge g(n)$, then $V_r-1>2Q^r$ (possible by Lemma 1). By part (i), we may assume that $g(n+1)$ is even; let $g(n+1)=2m$. Then, by (3),

$$a_{n+1}=V_{g(n+1)}=V_{2m}=V_m^2-2Q^m.$$

By Lemma 1, $2Q^m<V_m-1$ and, since $m\ge g(n), V_m\ge V_{g(n)}$, from which it follows that

$$a_{n+1}=V_m^2-2Q^m>V_m^2-V_m+1=V_m(V_m-1)+1>a_n^2-a_n+1.$$

This proves part (ii), by Theorem B.

Theorem 4: The series $\sum_{n=0}^{\infty} 1/V_{g(n)}$ is irrational if g is an integral-valued function such that $g(n+1) \geq 2g(n)+1 > 1$ for all sufficiently large n.

Proof: Assume that $g(n+1) \geq 2g(n)+1 > 1$ for all $n >$ some integer $N > 1$, and let $a_n = V_{g(n)}$. If $n > N$ and $g(n+1)$ is odd, then $a_{n+1} > a_n^2$ by Theorem 3. Assume $g(n+1)$ is even and let $g(n+1) = 2m$; then, since $m \geq g(n)+1/2$ is an integer, $m \geq g(n)+1$, i.e., $m-1 \geq g(n)$. By Lemma 2,

$$a_{n+1} = V_{g(n+1)} = V_{2m} > V_{2m-1} = V_{2(m-1)+1} > V_{m-1}^2 \geq V_{g(n)}^2 = a_n^2,$$

proving the result by Theorem A.

Examples: Since $F_n = U_n(1, -1)$, it is apparent that

$$\sum_{n=0}^{\infty} 1/F_{2^n}, \quad \sum_{n=0}^{\infty} 1/F_{2^n k}, \quad \text{and} \quad \sum_{n=0}^{\infty} 1/F_{2^n+1}$$

are special cases of Theorem 1. Other examples of series whose sum is irrational are

$$\sum_{n=0}^{\infty} 1/U_{cb^n} \;\; (c \geq 1 \text{ and } b \geq 2) \quad \text{and} \quad \sum_{n=0}^{\infty} 1/U_{2^n-k}, \; k \geq -1.$$

In fact, it is readily seen that, for $\{U_n\}$ any Lucas sequence, $\sum_{n=0}^{\infty} 1/U_{g(n)}$ is irrational if $g(n) = cb^n - f(n)$, where $c \geq 1$, $b \geq 2$, and f is an integer-valued function such that $f(n+1) \leq 2f(n)$ for all large n, provided $g(n) > 1$ for all large n (f could be, for example, any polynomial in n with positive leading coefficient). Similar examples illustrating Theorems 3 and 4 are readily obtained.

It is interesting that the sum of the series $\sum_{n=0}^{\infty} 1/U_{2^n r}$, $r \geq 1$, found by Lucas for $Q = \pm 1$ is not known for any other value of Q. Also, the sum of $\sum_{n=0}^{\infty} 1/V_{2^n r}$ is not known (however, see [1]), for any value of Q, nor is the sum of any of the other series whose irrationality we have shown in this paper.

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