A JUXTAPOSITION PROPERTY FOR THE 4 × 4 MAGIC SQUARE

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Consider the standard 4×4 magic square:

$$M = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}.$$

Now (using decimal notation) let us "juxtapose" the entries of the first two columns to form a single column of numbers and then compute the sum. Similarly, juxtapose the last two columns and consider that sum:

C1&C2	C3&C4
162	313
511	108
97	612
414	151
1184	1184

For the first operation here, we juxtapose Column 1 with Column 2, which we indicate by writing C1&C2. The other operation above is C3&C4. Curiously, these sums are equal.

Similarly, we can combine the other pairs of columns, with the extra condition that the entry "9" is viewed as the 2digit number "09":

C2&C1	C4&C3	C1&C3	C3&C1	C2&C4	C4&C2
216	133	163	316	213	132
115	810	510	105	118	811
709	126	96	609	712	127
144	115	415	154	141	114
1184	1184	1184	1184	1184	1184

Repeating this process with the other four possible choices, we obtain a different set of equal sums:

C1&C4	C4&C1	C2&C3	C3&C2
1613	1316	23	32
58	85	1110	1011
912	1209	76	67
41	14	1415	1514
2624	2624	2624	2624

Performing similar operations with the rows of M, we find that the sums do not behave quite so nicely, but still there are a number of equalities. For example:

R1&R2	R4&R3	R1&R4	R4&R1
165	409	164	416
211	147	214	142
310	156	315	153
138	112	131	113
824	824	824	824

The entry "9" was considered as a 2-digit number "09" throughout these operations. That was done to make the patterns of 1-digit numbers and 2-digit numbers in the square suitably symmetric. One possible way to avoid that device is to rewrite the square using "base nine" notation rather than the usual "base ten." The square can be written out in base nine as follows:

$$M_9 = \begin{bmatrix} 17 & 2 & 3 & 14 \\ 5 & 12 & 11 & 8 \\ 10 & 7 & 6 & 13 \\ 4 & 15 & 16 & 1 \end{bmatrix}.$$

This is the same square as the original, except that the numbers are written in another notation. Here, for example, $17_{\text{nine}} = 1*9^1 + 7*9^0 = 16_{\text{ten}}$. Then we note that the "juxtaposition property" works very well for M_9, since the 2-digit numbers are symmetrically distributed in this square. For example:

C1&C2	C3&C4
172	314
512	118
107	613
415	161
1317	1317
(Base nine)	(Base nine)

Here the numbers and additions are all done in base nine. For example "1317" equals 988 in base ten notation. The juxtaposition property can be shown to work just as well when base nine is used throughout the process.

It can be shown that the juxtaposition property is a consequence of the well-known 2×2 magic properties of M and the symmetries in the number of digits of the entries of M. According to the 2×2 magic property, M can be partitioned into four 2×2 squares, and the sum of the entries in each of these squares is again the magic constant 34. For example, the upper left corner is $\left[\begin{smallmatrix} 16 & 2 \\ 5 & 11 \end{smallmatrix}\right]$, which has the sum $16+2+5+11=34$. The same holds for all the corner squares, the inner central square, and the square formed by the four corner entries of M. If the inner columns are interchanged, or the inner two rows are interchanged, or if both the operations are performed together, the sum of the entries of all these squares remains 34. We can use these 2×2 properties to "explain" the patterns found on juxtaposition.

Consider the sum of the rows R1 and R3 to be R1 + R3 = (a, b, c, d). Since the sum of all four rows is (34, 34, 34, 34), we see that R2 + R4 = (a', b', c', d'), where $a' = 34 - a$, $b' = 34 - b$, $c' = 34 - c$, and $d' = 34 - d$. Note that $a + b$ is the sum of the entries of the upper left 2×2 square after interchanging the middle two rows and therefore equals 34; similarly, $c + d = a + c = b + d = 34$. Therefore, we see that

$$\text{R1} + \text{R3} = (a, a', a', a) \quad \text{and} \quad \text{R2} + \text{R4} = (a', a, a, a'), \tag{1}$$

where $a + a' = 34$.

For M, $a = 25$ and $a' = 9$. For M_9, $a = 27$ and $a' = 9$. Now the juxtaposition sum Cn&Cm can be written as

$$Cn\&Cm = 10^{d_{1m}} M_{1n} + M_{1m} + 10^{d_{2m}} M_{2n} + M_{2m} + 10^{d_{3m}} M_{3n} + M_{3m} + 10^{d_{4m}} M_{4n} + M_{4m}.$$

Here M_{kL} are the entries of the magic square in matrix notation, and d_{km} is the number of (base ten) digits in the number M_{km}. Since $M_{1m} + M_{2m} + M_{3m} + M_{4m} = 34$,

$$Cn\&Cm = 10^{d_{1m}} M_{1n} + 10^{d_{2m}} M_{2n} + 10^{d_{3m}} M_{3n} + 10^{d_{4m}} M_{4n} + 34.$$

Now certain symmetries can be observed in M (with 09 instead of 9):

$$d_{1m} = d_{3m} \quad \text{and} \quad d_{2m} = d_{4m} \quad \text{for all } m.$$

Therefore,

$$Cn\&Cm = 10\left(10^{(d_{1m}-1)}(\mathrm{R}1+\mathrm{R}3)_n + 10^{(d_{2m}-1)}(\mathrm{R}2+\mathrm{R}4)_n\right) + 34,$$

where the subscript n denotes the n^{th} element of the row sum.

Using the 2×2 magic properties (1),

$$Cn\&Cm = 10\left[10^{(d_{1m}-1)}a + 10^{(d_{2m}-1)}a'\right] + 34 \text{ for } n = 1 \text{ or } 4, \text{ and}$$

$$Cn\&Cm = 10\left[10^{(d_{1m}-1)}a' + 10^{(d_{2m}-1)}a\right] + 34 \text{ for } n = 2 \text{ or } 3.$$

It can be seen from M that:

(1) For $n = 1$ or 4 and $m = 2$ or 3, $d_{1m} = 1$ and $d_{2m} = 2$. Therefore,

$$C1\&C2 = C1\&C3 = C4\&C2 = C4\&C3 = 10(a + 10a') + 34.$$

(2) For $n = 2$ or 3 and $m = 1$ or 4, $d_{1m} = 2$ and $d_{2m} = 1$. Therefore,

$$\begin{aligned} C2\&C1 = C2\&C4 = C3\&C1 = C3\&C4 &= 10(a + 10a') + 34 \\ &= C1\&C2 = C1\&C3 = C4\&C2 = C4\&C3. \end{aligned}$$

Since $a = 25$ and $a' = 9$, all the above are equal to 1184.

(3) For $n = 1$ and $m = 4$, and for $n = 4$ and $m = 1$, $d_{1m} = 2$ and $d_{2m} = 1$.

(4) For $n = 2$ and $m = 3$, and for $n = 3$ and $m = 2$, $d_{1m} = 1$ and $d_{2m} = 2$.

For all these cases, the juxtaposition sums turn out to be equal to $10(a' + 10a) + 34 = 2624$.

From the 2×2 magic properties similar behaviors to equation (1) can be found for columns also. However, the pattern of 1- and 2-digit numbers needed for the equalities of juxtaposition sums do not match up so nicely for the row juxtapositions as for the column juxtapositions. Therefore, the number of equalities are less for the former.

All the above relations hold for bases other than ten, provided the symmetries in the number of digits in the entries are satisfied.

In summary, we find that these "juxtaposition properties" of the 4×4 magic square can be seen as some of the well-known internal symmetries of M.

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