A NOTE ON A GENERAL CLASS OF POLYNOMIALS

Richard André-Jeannin
IUT GEA, Route de Romain, 54400 Longwy, France
(Submitted April 1993)

1. INTRODUCTION

We consider polynomials $\{U_n(p,q;\ x)\}$ such that

$$U_n(p,q;\ x) = (x+p)U_{n-1}(p,q;\ x) - qU_{n-2}(p,q;\ x),\ n \geq 2 \tag{1}$$

with $U_0(p,q;\ x) = 0$ and $U_1(p,q;\ x) = 1$.

The parameters p and q are arbitrary real numbers (with $q \neq 0$), and we denote by $\alpha,\ \beta$ the numbers such that $\alpha + \beta = p$ and $\alpha\beta = q$.

We see by induction that there exists a sequence $\{c_{n,k}(p,q)\}_{\substack{n\geq 0\\ k\geq 0}}$ of numbers such that

$$U_{n+1}(p,q;\ x) = \sum_{k\geq 0} c_{n,k}(p,q)x^k, \tag{2}$$

with

$$c_{n,k}(p,q) = 0 \text{ if } k > n \quad\text{and}\quad c_{n,n}(p,q) = 1,\ n \geq 0.$$

The first few terms of the sequence $\{U_n(p,q;\ x)\}$ are

$$\begin{cases} U_2(p,q;\ x) = p + x \\ U_3(p,q;\ x) = (p^2 - q) + 2px + x^2 \\ U_4(p,q;\ x) = (p^3 - 2pq) + (3p^2 - 2q)x + 3px^2 + x^3. \end{cases}$$

Particular cases of $U_n(p,q;\ x)$ are the Fibonacci polynomials $F_n(x)$, the Pell polynomials $P_n(x)$ [4], the first Fermat polynomials $\Phi_n(x)$ [5], the Morgan–Voyce polynomials of the second kind $B_n(x)$ ([3], [6], [8], [9]), and the Chebyschev polynomials of the second kind $S_n(x)$ given by

$$U_n(0,-1;\ x) = F_n(x),$$
$$U_n(0,-1;\ 2x) = P_n(x),$$
$$U_n(0,2;\ x) = \Phi_n(x),$$
$$U_{n+1}(2,1;\ x) = B_n(x),$$
$$U_n(0,1;\ 2x) = S_n(x).$$

We have used S_n in place of the customary U_n since U_n has been used in a different way in the present paper. For particular values of the variable x, one can obtain some interesting sequences of numbers.

(i) The sequence $\{U_n(p,q;\ -p)\}$ satisfies the recurrence

$$U_n(p,q;\ -p) = -qU_{n-2}(p,q;\ -p),\ n \geq 2;$$

thus,

$$U_{2n}(p,q;\ -p) = 0 \quad\text{and}\quad U_{2n+1}(p,q;\ -p) = (-q)^n.$$

By (2), these can also be written

$$\sum_{k=0}^{2n-1}(-1)^k p^k c_{2n-1,k}(p,q)=0 \tag{3}$$

and

$$\sum_{k=0}^{2n}(-1)^k p^k c_{2n,k}(p,q)=(-1)^n q^n. \tag{4}$$

(ii) It follows at once that the sequence $\{U_n(p,q;\ 0)\}$ is the generalized Fibonacci sequence defined by

$$U_n(p,q;\ 0)=pU_{n-1}(p,q;\ 0)-qU_{n-2}(p,q;\ 0),$$

with $U_0(p,q;\ 0)=0$ and $U_1(p,q;\ 0)=1$. Therefore,

$$U_{n+1}(p,q;\ 0)=\sum_{i+j=n}\alpha^i\beta^j=\begin{cases}\dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} & \text{if } \alpha\neq\beta,\\ (n+1)\alpha^n & \text{if } \alpha=\beta.\end{cases}$$

By (2), notice that

$$c_{n,0}(p,q)=U_{n+1}(p,q;\ 0)=\sum_{i+j=n}\alpha^i\beta^j. \tag{5}$$

More generally, our aim is to express the coefficient $c_{n,k}(p,q)$ as a polynomials in (α,β) and as a polynomial in (p,q).

2. THE TRIANGLE OF COEFFICIENTS

One can display the sequence $\{c_{n,k}(p,q)\}$ in a triangle, thus:

$n \backslash k$	0	1	2	3	...
0	1	0	0	0	
1	p	1	0	0	
2	p^2-q	$2p$	1	0	
3	p^3-2pq	$3p^2-2q$	$3p$	1	
⋮					

Comparing the coefficients of x^k in the two members of (1), we see by (2) that, for $n\geq 2$ and $k\geq 1$,

$$\begin{aligned}c_{n,k}(p,q)&=c_{n-1,k-1}(p,q)+pc_{n-1,k}(p,q)-qc_{n-2,k}(p,q)\\&=c_{n-1,k-1}+\beta c_{n-1,k}+\alpha(c_{n-1,k}-\beta c_{n-2,k})\\&=c_{n-1,k-1}+\alpha c_{n-1,k}+\beta(c_{n-1,k}-\alpha c_{n-1,k}),\end{aligned} \tag{6}$$

where, for brevity, we put $c_{n,k}$ for $c_{n,k}(p,q)$. From this, one can easily obtain another recurrence relation.

Theorem 1: For every $n \geq 1$ and $k \geq 1$, we have

$$\begin{aligned} c_{n,k} &= \beta c_{n-1,k} + \sum_{i=0}^{n-1} \alpha^{n-1-i} c_{i,k-1} \\ &= \alpha c_{n-1,k} + \sum_{i=0}^{n-1} \beta^{n-1-i} c_{i,k-1}. \end{aligned} \tag{7}$$

Proof: In fact, (7) is clear by direct computation for $n \leq 2$ (recall that $\alpha + \beta = p$). Supposing that the relation is true for $n \geq 2$, then we have by (6) that

$$\begin{aligned} c_{n+1,k} &= \beta c_{n,k} + \alpha(c_{n,k} - \beta c_{n-1,k}) + c_{n,k-1} \\ &= \beta c_{n,k} + \alpha \sum_{i=0}^{n-1} \alpha^{n-1-i} c_{i,k-1} + c_{n,k-1} \\ &= \beta c_{n,k} + \sum_{i=0}^{n} \alpha^{n-i} c_{i,k-1}. \end{aligned}$$

This concludes the proof, and the other formula can be proved in the same way.

Let us examine some particular cases.

(i) Fibonacci polynomials. In this case we have $p = 0, q = -1$, and $\alpha = -\beta = 1$. From this, (7) becomes

$$\begin{aligned} c_{n,k} &= -c_{n-1,k} + \sum_{i=0}^{n-1} c_{i,k-1} \\ &= c_{n-1,k} + \sum_{i=0}^{n-1} (-1)^{n-1-i} c_{i,k-1}. \end{aligned}$$

(ii) Morgan–Voyce polynomials of the second kind. In this case, we have $p = 2, q = 1$, and $\alpha = \beta = 1$. Thus, (7) becomes

$$c_{n,k} = c_{n-1,k} + \sum_{i=0}^{n-1} c_{i,k-1},$$

which is the recursive definition of the DFFz triangle [2], known to be the triangle of coefficients of Morgan–Voyce polynomials ([1], [3]).

3. DETERMINATION OF $c_{n,k}(p, q)$ AS A POLYNOMIAL IN (α, β)

In our proof we shall need the following lemma.

Lemma: For every $k \geq 0$, we have

$$\frac{1}{(1 - pt + qt^2)^{k+1}} = \sum_{n \geq 0} d_{n,k} t^n, \tag{8}$$

with

$$d_{n,k} = \sum_{i+j=n} \binom{k+i}{k} \binom{k+j}{k} \alpha^i \beta^j.$$

Proof: Recall that

$$\phi_r(t) = \frac{1}{(1-rt)^{k+1}} = \sum_{n\geq 0}\binom{k+n}{k} r^n t^n,$$

where r is a real or complex parameter and $|rt|<1$. Thus, we have

$$\frac{1}{(1-pt+qt^2)^{k+1}} = \frac{1}{(1-\alpha t)^{k+1}(1-\beta t)^{k+1}}$$

$$= \sum_{n\geq 0}\binom{k+n}{k}\alpha^n t^n \cdot \sum_{n\geq 0}\binom{k+n}{k}\beta^n t^n$$

$$= \sum_{n\geq 0} d_{n,k} t^n,$$

where

$$d_{n,k} = \sum_{i+j=n}\binom{k+i}{k}\binom{k+j}{k}\alpha^i\beta^j,$$

by application of Cauchy's rule for multiplying power series. Q.E.D.

Theorem 2: For every $n \geq 0$ and $k \geq 0$, we have

$$c_{n,k}(p,q) = \sum_{i+j=n-k}\binom{k+i}{k}\binom{k+j}{k}\alpha^i\beta^j, \tag{9}$$

where we have used the convention $\sum_{i+j=s} a_{i,j} = 0$, if $s<0$.

Proof: For brevity, we put $U_n(p,q;\ x) = U_n(x)$ and $c_{n,k}(p,q) = c_{n,k}$. Let us define the generating function of the sequence $\{U_n(x)\}$ by

$$f(x,t) = \sum_{n\geq 0} U_{n+1}(x)t^n.$$

By (1), we get

$$f(x,t)-1 = \sum_{n\geq 1} U_{n+1}(x)t^n = t(x+p)\sum_{n\geq 1} U_n(x)t^{n-1} - qt^2\sum_{n\geq 1} U_{n-1}(x)t^{n-2}.$$

The last sum can be written as $\sum_{n\geq 2} U_{n-2}(x)t^{n-2}$, since $U_0(x)=0$. It follows from this that

$$f(x,t)-1 = t(x+p)f(x,t) - qt^2 f(x,t).$$

Thus,

$$f(x,t) = \frac{1}{1-(x+p)t+qt^2}. \tag{10}$$

We deduce from (10) that

$$\frac{k!\,t^k}{(1-(x+p)t+qt^2)^{k+1}} = \frac{\partial^k}{\partial x^k} f(x,t) = \sum_{n\geq 0} U_{n+1}^{(k)}(x)t^n$$

$$= \sum_{n\geq k} U_{n+1}^{(k)}(x)t^n = \sum_{n\geq 0} U_{n+k+1}^{(k)}(x)t^{n+k},$$

since $U_{n+1}(x)$ is a polynomial of degree n.

Put $x = 0$ in the last formula and recall that

$$c_{n+k,k} = \frac{U_{n+k+1}^{(k)}(0)}{k!},$$

by Taylor's formula, to obtain

$$\frac{1}{(1-pt+qt^2)^{k+1}} = \sum_{n\geq 0} c_{n+k,k} t^n. \tag{11}$$

Comparing this formula with (8), we see that

$$c_{n+k,k} = d_{n,k} = \sum_{i+j=n} \binom{k+i}{k}\binom{k+j}{k} \alpha^i \beta^j.$$

This concludes the proof.

Remarks: ***(i)*** If $k = 0$, then (9) reduces to the classical formula (5).

(ii) Notice that (11) is the generating function of the k^{th} column of the triangle of coefficients $c_{n,k}$. If $k = 0$, we obtain in particular the well-known generating function of the generalized Fibonacci sequence, namely,

$$\frac{1}{1-pt+qt^2} = \sum_{n\geq 0} U_{n+1}(p, q;\ 0)t^n. \tag{12}$$

(iii) Using (6), one can obtain, by induction and with a little manipulation, another proof of Theorem 2.

Corollary 1: For every $n \geq 0$ and $k \geq 0$, we have

$$c_{n,k}(-p, q) = (-1)^{n-k} c_{n,k}(p, q).$$

Proof: The result follows immediately from (9) and the fact that $(-\alpha)+(-\beta) = -p$ and $(-\alpha)(-\beta) = q$.

4. SOME PARTICULAR CASES

The general formula (9) can be simplified in two cases:

(i) Supposing that $p^2 = 4q$, we have $\alpha = \beta$ and (8) becomes

$$\frac{1}{(1-pt+qt^2)^{k+1}} = \frac{1}{(1-\alpha t)^{2k+2}} = \sum_{n\geq 0} \binom{n+2k+1}{2k+1} \alpha^n t^n.$$

Hence, by (11), $c_{n,k} = c_{n,k}(p, q)$ takes the simpler form

$$c_{n,k} = \binom{n+k+1}{2k+1} \alpha^{n-k} = \binom{n+k+1}{2k+1} (p/2)^{n-k}.$$

If $p = 2$ and $q = 1$ (Morgan–Voyce polynomials of the second kind), we obtain the known relation [8]

$$B_n(x) = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} x^n.$$

(ii) Supposing that $p = 0$, we have $\alpha = -\beta$ and (8) becomes

$$\frac{1}{(1-pt+qt^2)^{k+1}} = \frac{1}{(1+qt^2)^{k+1}} = \sum_{n\geq 0} (-1)^n \binom{n+k}{k} q^n t^{2n}.$$

Thus, by (11),

$$c_{2n+k,\,k} = (-1)^n \binom{n+k}{n} q^n \quad \text{and} \quad c_{2n+k+1,\,k} = 0 \ \text{ for } n \geq 0 \text{ and } k \geq 0.$$

This can be written

$$c_{2k+n,\,n} = (-1)^k \binom{k+n}{k} q^k \quad \text{and} \quad c_{2k+n+1,\,n} = 0.$$

Hence,

$$c_{n,\,n-2k} = (-1)^k \binom{n-k}{k} q^k, \ \text{ for } n-2k \geq 0 \ \text{ and } \ c_{n,\,n-2k-1} = 0, \ \text{ for } n-2k-1 \geq 0.$$

Now, by (2),

$$U_{n+1}(0, q;\, x) = \sum_{k=0}^{n} c_{n,\,k}(0, q) x^k = \sum_{k=0}^{n} c_{n,\,n-k}(0, q) x^{n-k} = \sum_{k=0}^{[n/2]} c_{n,\,n-2k}(0, q) x^{n-2k}.$$

Thus, we get the simplified formula

$$U_{n+1}(0, q;\, x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} q^k x^{n-2k}. \tag{13}$$

If $p = 0$ and $q = -1$, we obtain the known decomposition of Fibonacci polynomials

$$F_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k},$$

and if $p = 0$ and $q = 1$, we have the similar expression of Chebyschev polynomials of the second kind

$$S_{n+1}(x) = U_{n+1}(0, 1;\, 2x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

5. DETERMINATION OF $c_{n,k}(p,q)$ AS A POLYNOMIAL IN (p,q)

Theorem 3: For every $n \geq 0$ and $k \geq 0$, we have

$$c_{n,\,k}(p, q) = \sum_{r=0}^{[(n-k)/2]} (-1)^r \binom{n-r}{r} \binom{n-2r}{k} q^r p^{n-2r-k}. \tag{14}$$

Proof: It is clear that $U_{n+1}(p, q;\, x) = U_{n+1}(0, q;\, x+p)$. Thus,

$$c_{n,\,k}(p, q) = \frac{U_{n+1}^{(k)}(p, q;\, 0)}{k!} = \frac{U_{n+1}^{(k)}(0, q;\, p)}{k!}.$$

By (13), one can express the last member as

$$\sum_{r=0}^{[n/2]}(-1)^r\binom{n-r}{r}\frac{(n-2r)\cdots(n-2r-k+1)}{k!}q^r p^{n-2r-k}$$
$$=\sum_{r=0}^{[(n-k)/2]}(-1)^r\binom{n-r}{r}\binom{n-2r}{k}q^r p^{n-2r-k}$$

This completes the proof of Theorem 3.

If $k = 0$, we get the formula known by Lucas ([7], p. 207), namely,

$$U_{n+1}(p,q;\ 0)=\sum_{r=0}^{[n/2]}(-1)^r\binom{n-r}{r}q^r p^{n-2r}. \tag{15}$$

6. RISING DIAGONAL FUNCTIONS

Let us define the rising diagonal functions $\{\Psi_n(p,q;\ x)\}$ of the sequence $\{c_{n,k}(p,q)\}$—see the table—by $\Psi_0(p,q;\ x)=0$ and

$$\Psi_{n+1}(p,q;\ x)=\sum_{k=0}^{n}c_{n-k,k}(p,q)x^k=\sum_{k=0}^{[n/2]}c_{n-k,k}(p,q)x^k,\ \text{for } n\geq 0. \tag{16}$$

Notice that, from the table,

$$\Psi_1(p,q;\ x)=1,\ \Psi_2(p,q;\ x)=p,\ \text{and}\ \Psi_3(p,q;\ x)=p^2-q+x. \tag{17}$$

Theorem 4: For every $n\geq 2$, we have

$$\Psi_n(p,q;\ x)=p\Psi_{n-1}(p,q;\ x)+(x-q)\Psi_{n-2}(p,q;\ x). \tag{18}$$

Proof: For brevity, we put $\Psi_n(p,q;\ x)=\Psi_n(x)$ and $c_{n,k}(p,q)=c_{n,k}$. By (17), the statement holds for $n=2$ and $n=3$. Supposing that (18) is true for $n\geq 3$, then we get, by (16),

$$\Psi_{n+1}(x)=c_{n,0}+\sum_{k=1}^{[n/2]}c_{n-k,k}x^k.$$

Recall from (5) that $c_{n,0}=U_{n+1}(0)=pc_{n-1,0}-qc_{n-2,0}$, and notice that $n-k\geq n-[n/2]\geq 2$, since $n\geq 3$. By these remarks and (6), one can write

$$\Psi_{n+1}(x)=pc_{n-1,0}-qc_{n-2,0}+\sum_{k=1}^{[n/2]}(c_{n-1-k,k-1}+pc_{n-1-k,k}-qc_{n-2-k,k})x^k$$
$$=p\sum_{k=0}^{[n/2]}c_{n-1-k,k}x^k-q\sum_{k=0}^{[n/2]}c_{n-2-k,k}x^k+x\sum_{k=0}^{[n/2]-1}c_{n-2-k,k}x^k$$
$$=p\Psi_n(x)+(x-q)\Psi_{n-1}(x),\ \text{since } [n/2]-1=[(n-2)/2].$$

This concludes the proof.

Corollary 2: For every $n \geq 0$, we have

$$\Psi_{n+1}(p, q;\ x) = \sum_{r=0}^{[n/2]} \binom{n-r}{r} p^{n-2r}(x-q)^r. \tag{19}$$

Proof: By Theorem 4, and since $\Psi_0(x) = 0$, $\Psi_1(x) = 1$, it is clear that

$$\Psi_n(p, q;\ x) = U_n(p, q-x;\ 0),$$

and the result follows by (15).

Let us examine some particular cases.

(i) Put $x = q$ in (19) to get, by (16),

$$\sum_{k=0}^{[n/2]} q^k c_{n-k,k}(p, q) = p^n.$$

If $p = 1$ and $q = 1$, we get a known identity on the coefficients of the Morgan–Voyce polynomial of the second kind B_n, first noticed by Ferri, Faccio and D'Amico ([2], [3]), namely,

$$\sum_{k=0}^{[n/2]} c_{n-k,k}(2, 1) = 2^n.$$

(ii) Put $x = 1$ in (19) to get, by (16),

$$\sum_{k=0}^{[n/2]} c_{n-k,k}(p, q) = \sum_{r=0}^{[n/2]} \binom{n-r}{r} p^{n-2r}(1-q)^r,$$

which is more general than the above result.

(iii) If $p = 0$, then Corollary 2 implies by (16) that

$$\sum_{k=0}^{n} c_{2n-k,k}(0, q)x^k = (x-q)^n.$$

If $q = 1$ (Chebyschev polynomials of the second kind), or $q = 2$ (first Fermat polynomials), this identity was first noticed by Horadam [5] with slightly different notations.

7. THE ORTHOGONALITY OF THE SEQUENCE $\{U_n(p, q;\ x)\}$

In this paragraph we shall suppose that $q > 0$. Consider the sequence $\{R_n(p, q;\ x)\}$ defined by

$$R_n(p, q;\ x) = q^{(n-1)/2} S_n\left(\frac{x+p}{2\sqrt{q}}\right), \tag{20}$$

where $S_n(x)$ is the n^{th} Chebyschev polynomial of the second kind. Let us determine the recurrence satisfied by the sequence $\{R_n(p, q;\ x)\}$. One can write

$$R_n(p, q;\ x) = q^{(n-1)/2}\left[\left(\frac{x+p}{\sqrt{q}}\right) S_{n-1}\left(\frac{x+p}{2\sqrt{q}}\right) - S_{n-2}\left(\frac{x+p}{2\sqrt{q}}\right)\right]$$

$$= (x+p)1^{(n-2)/2} S_{n-1}\left(\frac{x+p}{2\sqrt{q}}\right) - q\,q^{(n-3)/2} S_{n-2}\left(\frac{x+p}{2\sqrt{q}}\right)$$

$$= (x+p)R_{n-1}(p,q;\ x) - qR_{n-2}(p,q;\ x).$$

Observe that the sequence $\{R_n(p,q;\ x)\}$ satisfies the recurrence (1) with $R_0(p,q;\ x) = 0$ and $R_1(p,q;\ x) = 1$, so that

$$R_n(p,q;\ x) = U_n(p,q;\ x). \tag{21}$$

Recalling that the sequence $\{S_n(x)\}$ is orthogonal over $[-1,1]$ with respect to the weight $\sqrt{1-x^2}$, we deduce that the sequence $\{U_n(p,q;\ x)\}$ is orthogonal over $[-p-2\sqrt{q}, -p+2\sqrt{q}]$ with respect to the weight $w(x) = \sqrt{-x^2-2px-\Delta}$, where $\Delta = p^2-4q$.

In fact, for $n \neq m$, we have

$$\int_{-p-2\sqrt{q}}^{-p+2\sqrt{q}} U_n(x)U_m(x)w(x)\,dx = q^{((n+m)/2)-1}\int_{-p-2\sqrt{q}}^{-p+2\sqrt{q}} S_n\left(\frac{x+p}{2\sqrt{q}}\right)S_m\left(\frac{x+p}{2\sqrt{q}}\right)w(x)\,dx$$

$$= 4q^{(n+m)/2}\int_{-1}^{+1} S_n(\omega)S_m(\omega)\sqrt{1-\omega^2}\,d\omega = 0,$$

where $\omega = \frac{x+p}{2\sqrt{q}}$. In the case of the Morgan–Voyce polynomial of the second kind, $B_n(x)$, this orthogonality result was first given by Swamy [8].

If $\omega = \cos t$ $(0 < t < \pi)$, it is well known that $S_n(\omega) = \frac{\sin nt}{\sin t}$, Thus, by (20) and (21), we have

$$U_n(p,q;\ -p+2\omega\sqrt{q}) = q^{(n-1)/2}S_n(\omega) = q^{(n-1)/2}\frac{\sin nt}{\sin t}.$$

From this, we see that the roots of $U_n(p,q;\ x)$ are given by

$$x_k = -p+2\sqrt{q}\cos(k\pi/n),\ k = 1,\ldots,(n-1).$$

For instance, the roots of the Morgan–Voyce polynomial of the second kind, $B_n(x) = U_{n+1}(2,1;\ x)$, are (see [9])

$$x_k = -2+2\cos\left(\frac{kn}{n+1}\right) = -4\sin^2\left(\frac{k\pi}{2n+2}\right),\quad k = 1,\ldots,(n-1).$$

Under the hypothesis $q > 0$, we deduce from the general expression for x_k that the generalized Fibonacci sequences $U_n(p,q;\ 0)$ vanish if and only if there exists an integer k $(1 \le k \le n-1)$ such that $\cos(k\pi/n) = p/2\sqrt{q}$.

8. CONCLUDING REMARK

In a future paper, we shall investigate the sequence $\{V_n(p,q;\ x)\}$ of polynomials, defined by

$$V_n(p,q;\ x) = (x+p)V_{n-1}(p,q;\ x) - qV_{n-2}(p,q;\ x),\ n \ge 2,$$

with $V_0(p,q;\ x) = 2$ and $V_1(p,q;\ x) = x+p$.

ACKNOWLEDGMENT

The author wishes to thank the referee for bringing to his attention Horadam's papers on polynomials. I appreciate, equally, his or her helpful comments for improving the paper.

REFERENCES

1. R. André-Jeannin. "A Generalization of Morgan–Voyce Polynomials." *The Fibonacci Quarterly* (to appear).
2. G. Ferri, M. Faccio, & A. D'Amico. "Fibonacci Numbers and Ladder Network Impedance." *The Fibonacci Quarterly* **30.1** (1992):62-67.
3. G. Ferri, M. Faccio, & A. D'Amico. "The DFF and DFFz Triangles and Their Mathematical Properties." *Proceedings of the Fifth International Conference on Fibonacci Numbers and Their Applications*. (to appear).
4. A. F. Horadam & Br. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.2** (1985):7-20.
5. A. F. Horadam. "Chebyschev and Fermat Polynomials for Diagonal Functions." *The Fibonacci Quarterly* **19.4** (1979):328-333.
6. J. Lahr. "Fibonacci and Lucas Numbers and the Morgan–Voyce Polynomials in Ladder Networks and in Electrical Line Theory." In *Fibonacci Numbers and Their Applications* **1:**141-61. Ed. A. N. Philippou, G. E. Bergum, & A. F. Horadam. Dordrecht: Kluwer, 1986.
7. E. Lucas. "Théorie des fonctions numériques simplement périodiques." *Amer. J. Math.* **1** (1878):184-220, 289-321.
8. M. N. S. Swamy. "Properties of the Polynomials Defined by Morgan–Voyce." *The Fibonacci Quarterly* **4.1** (1966):73-81.
9. M. N. S. Swamy. "Further Properties of Morgan–Voyce Polynomials." *The Fibonacci Quarterly* **6.2** (1968):167-75.

AMS Classification Numbers: 11B39, 33C25

❖❖❖