THE FIBONACCI KILLER*

Peter J. Grabner
Department of Mathematics, Technical University of Graz, Austria

Helmut Prodinger
Department of Algebra and Discrete Mathematics, Technical University of Vienna, Austria
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1. INTRODUCTION

We consider the following stochastic process: Assume that a "player" is hit at any time x with probability p. However, he dies only after two consecutive hits. We might code this process by **0** and **1**, marking a hit, e.g., by a "**1**". Then the sequences associated with a player can be described by

$$\{\mathbf{0}, \mathbf{10}\}^* \cdot \mathbf{11}.$$

The notation $\{\mathbf{0}, \mathbf{10}\}^*$ denotes arbitrary sequences consisting of the blocks **0** and **10**, the block **11** are the fatal hits. Notice that $\{\mathbf{0}, \mathbf{10}\}^*$ are exactly the admissible blocks in the Fibonacci expansion of integers (*Zeckendorf* expansion, cf. [13]). Accordingly, the generating function

$$\frac{p^2z^2}{1-qz-pqz^2} \tag{1.1}$$

has as the coefficient of z^x the probability $\mathbb{P}\{X=x\}$ that the lifetime X of a player is exactly x. The generating function (1.1) is known in the context of the Fibonacci distribution or geometric distribution of order 2, cf. [1], [3], [4], [7], [8], [10], [12].

Here, we are interested in n (independent) players subject to this game and ask when (in the sense of a mean value) the last player dies.

Without the "Fibonacci" restriction, i.e., the maximum of n (independent) geometric random variables, this problem has been studied previously and has some applications. (Compare [5], [11].)

We have obviously

$$\mathbb{P}\{\max\{X_1, \dots, X_n\} \le x\} = (\mathbb{P}\{X \le x\})^n. \tag{1.2}$$

The generating function of $\mathbb{P}\{X > x\}$ is given by

$$\frac{1+pz}{1-qz-pqz^2}.$$

We now factor the denominator of this function to obtain

$$1-qz-pqz^2 = (1-az)(1-bz)$$

with

$$a = \frac{q+\sqrt{q^2+4pq}}{2} \quad \text{and} \quad b = \frac{q-\sqrt{q^2+4pq}}{2}.$$

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Performing the partial fraction decomposition and extracting coefficients yields

$$\mathbb{P}\{X > x\} = \frac{1}{\sqrt{q^2+4pq}}(a^x(a+p) - b^x(b+p)).$$

Using (1.2) we obtain the expectation for the maximum lifetime of n players:

$$\mathbb{E}_n = \mathbb{E}\max\{X_1, \ldots, X_n\} = \sum_{x\geq 0}\left(1 - \left(1 - \frac{1}{\sqrt{q^2+4pq}}(a^x(a+p) - b^x(b+p))\right)^n\right). \tag{1.3}$$

By the binomial theorem we obtain

$$\mathbb{E}_n = \sum_{m=1}^{n}(-1)^{m-1}\binom{n}{m}\sum_{x\geq 0}(Aa^x - Bb^x)^m, \tag{1.4}$$

where we use the notation

$$A = \frac{a+p}{\sqrt{q^2+4pq}} = \frac{a^2}{q\sqrt{q^2+4pq}} \quad \text{and} \quad B = \frac{b+p}{\sqrt{q^2+4pq}} = \frac{b^2}{q\sqrt{q^2+4pq}}.$$

For example, in the symmetric case $p = q = \frac{1}{2}$, we have $a = \frac{1+\sqrt{5}}{4}$, $b = \frac{1-\sqrt{5}}{4}$, $A = \frac{5+3\sqrt{5}}{10}$, $B = \frac{5-3\sqrt{5}}{10}$.

We will find that $\mathbb{E}_n \sim \log_{1/a} n$ and refer for the (technical) proof and a more precise statement to the next section.

2. ASYMPTOTIC ANALYSIS

In (1.4) we found the expression

$$\mathbb{E}_n = \sum_{m=1}^{n}(-1)^{m-1}\binom{n}{m}f(m), \tag{2.1}$$

containing the function

$$f(z) = \sum_{x\geq 0}(Aa^x - Bb^x)^z \quad \text{for } \Re z > 0.$$

For an expression of that type we can write a complex contour integral

$$\mathbb{E}_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} f(z)\,dz, \tag{2.2}$$

where $\mathcal{C}$ is a positively oriented Jordan curve encircling the points 1, 2, ..., n (and no other integer points); this can easily be checked by residue calculus.

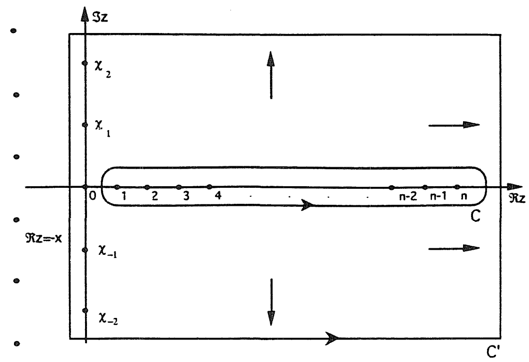
We will use Rice's method to obtain an asymptotic expansion for $\mathbb{E}_n$. For this we refer, e.g., to [2] and [6]. This method is based on a deformation of the contour of integration. For this purpose we need an analytic continuation of the function f to a region containing a half-plane $\Re z > -\varepsilon$ for $\varepsilon > 0$ (we actually give an analytic continuation to the whole complex plane).

Using the notation $C = B/A$ and $d = b/a$ (observe that $|C| < 1$ and $|d| < 1$) we obtain

$$\begin{aligned} f(z) &= A^z \sum_{x\geq 0} a^{xz}(1-Cd^x)^z = A^z \sum_{x\geq 0} a^{xz} \sum_{\ell\geq 0} (-1)^\ell C^\ell d^{x\ell} \binom{z}{\ell} \\ &= A^z \sum_{\ell\geq 0} (-1)^\ell C^\ell \binom{z}{\ell} \sum_{x\geq 0} (a^z d^\ell)^x = A^z \sum_{\ell\geq 0} \binom{z}{\ell} \frac{(-1)^\ell C^\ell}{1-a^z d^\ell} \end{aligned} \tag{2.3}$$

where the reversion of the order of summation was justified because of the absolute convergence of the sum for $\Re z > 0$. The sum in the last line gives a valid expression for $f(z)$ for every complex number z which is not a solution of any of the equations $1-a^z d^\ell = 0$. In the points $z_{\ell,x} = -\ell \frac{\log d}{\log a} + \frac{2x\pi i}{\log a}$ with $\ell = 0, 1, \ldots$ and $x \in \mathbb{Z}$, there are simple poles with residue

$$A^{z_{\ell,x}} \binom{z_{\ell,x}}{\ell} \frac{(-1)^{\ell-1} C^\ell}{\log a}.$$



The Contours of Integration

In order to be able to deform the contour of integration, we need an estimate for $f(z)$ along the vertical line $\Re z = -u$. For this purpose, we write

$$f(z) - \frac{A^z}{1-a^z} = \sum_{x\geq 0} A^z a^{xz}((1-Cd^x)^z - 1)$$

and observe the inequality $|(1-Cd^x)^z - 1| \leq \min(2, |z|Cd^x)$. This yields

$$\left| f(z) - \frac{A^z}{1-a^z} \right| \leq A^{-u} \left(\sum_{0\leq x\leq \log|z|} 2|a|^{-xu} + |z| \sum_{x>\log|z|} a^{-xu} C d^x \right) \ll |z|^\alpha \tag{2.4}$$

for $|d| < a^u < 1$ and $\alpha = -u \log a$.

We are now ready to start the deformation of the contour of integration: we take $\mathcal{C}'$ as the new contour and write

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} f(z)\,dz \\ &= \frac{1}{2\pi i} \oint_{\mathcal{C}'} \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} f(z)\,dz - \sum \operatorname*{Res}_{z=z_i} \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} f(z), \end{aligned} \tag{2.5}$$

Notice that there is a second-order pole at 0. Computation of residues yields (with $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$)

$$\operatorname*{Res}_{z=0} \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} f(z) = \frac{1}{\log a} H_n + \frac{\log A}{\log a} - \frac{1}{2},$$
$$\operatorname*{Res}_{z=\chi_x} \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} f(z) = \frac{A^{\chi_x}}{\chi_x \log a} \frac{n!\Gamma(1-\chi_x)}{\Gamma(n+1-\chi_x)} \quad \text{for } x \neq 0, \tag{2.6}$$

where $\chi_x = \frac{2x\pi i}{\log a} = z_{0,x}$.

Shifting the upper, the lower, and the right part of $\mathcal{C}'$ (cf. the figure) to infinity and observing that the integrals over these parts of the contour vanish then yields

$$\begin{aligned}\mathbb{E}_n = \frac{1}{\log\frac{1}{a}} H_n - \frac{\log A}{\log a} + \frac{1}{2} - \sum_{x\in\mathbb{Z}\setminus\{0\}} \frac{A^{\chi_x}}{\chi_x \log a} \frac{n!\Gamma(1-\chi_x)}{\Gamma(n+1-\chi_x)} \\ - \frac{1}{2\pi i} \int_{-u-i\infty}^{-u+i\infty} \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} f(z)\, dz.\end{aligned} \tag{2.7}$$

We now use the well-known asymptotic expansions

$$H_n = \log n + \gamma + O\left(\frac{1}{n}\right) \quad \text{and} \quad \frac{n!}{\Gamma(n+1-\chi_x)} = n^{\chi_x}\left(1+O\left(\frac{x^2}{n}\right)\right)$$

(by Stirling's formula) to formulate our main result.

Theorem 1: The expected maximal lifetime $\mathbb{E}_n$ of n independent players each of which has the Fibonacci distribution (or geometric distribution of order 2) fulfills, for $n \to \infty$,

$$\mathbb{E}_n = \log_{1/a} n - \frac{\gamma + \log A}{\log a} + \frac{1}{2} - \varphi(\log_{1/a} n) + O(n^{-u}), \tag{2.8}$$

for $0 < u < \min\left(1, \frac{\log|d|}{\log a}\right)$, and φ denotes a continuous periodic function of period 1 and mean 0 given by the Fourier expansion

$$\varphi(t) = \frac{1}{\log a} \sum_{x\in\mathbb{Z}\setminus\{0\}} A^{\chi_x}\Gamma(-\chi_x)e^{2x\pi i t} = \frac{1}{\log a} \sum_{x\in\mathbb{Z}\setminus\{0\}} \Gamma(-\chi_x)e^{2x\pi i(t-\log_{1/a} A)}, \tag{2.9}$$

which is rapidly convergent due to the exponential decay of the Γ-function along vertical lines. The remainder term is obtained by a trivial estimate of the integral and the (uniform) O-terms in Stirling's formula.

3. EXTENSIONS

Here, we briefly sketch the more general case where k consecutive hits are necessary to kill a player. In this case, the probability $\mathbb{P}(X = x)$ was derived by Philippou and Muwafi [9] in terms of multinomial coefficients. As described in the introduction, there is a bijection to the sequences

$$\{\mathbf{0}, \mathbf{10}, \mathbf{110}, \ldots, \mathbf{1}^{k-1}\mathbf{0}\} \cdot \mathbf{1}^k,$$

which yield the probability generating function

$$\frac{p^k z^k}{1-qz-pqz^2-\cdots-p^{k-1}qz^k}=\frac{p^k z^k(1-pz)}{1-z+qp^k z^{k+1}} \tag{3.1}$$

for the lifetime of a player (cf. [1, pp. 299ff], [3, p. 428], [7, p. 207], [8]). Likewise, the generating function of $\mathbb{P}\{X>x\}$ is given by

$$\frac{1-p^k z^k}{1-z+qp^k z^{k+1}}. \tag{3.2}$$

Again we factor the polynomial in the denominator

$$1-qz-pqz^2-\cdots p^{k-1}qz^k=(1-\alpha z)(1-\alpha_2 z)\cdots(1-\alpha_k z)$$

with $|\alpha|>|\alpha_2|\geq\cdots\geq|\alpha_k|$ $(\alpha>0)$. Then we have, by partial fraction decomposition and extracting coefficients,

$$\mathbb{P}\{X>x\}=A\alpha^x+A_2\alpha_2^x+\cdots+A_k\alpha_k^x \tag{3.3}$$

with $A=\frac{\alpha(\alpha-p)}{q((k+1)\alpha-k)}$ and similar expressions for $A_2,\ldots,A_k$.

For the expectation of the maximal lifetime of n players, we obtain

$$\mathbb{E}_{n,k}=\mathbb{E}\max\{X_1,\ldots,X_n\}=\sum_{m=1}^{n}(-1)^{m-1}\binom{n}{m}g(m)$$

with

$$g(z)=\sum_{\ell\geq 0}(A\alpha^\ell+\cdots+A_k\alpha_k^\ell)^z \text{ for } \Re z>0.$$

For the purpose of analytic continuation of g, we consider $g(z)-\frac{A^z}{1-\alpha^z}$ and proceed as in (2.4) to obtain the continuation and a polynomial estimate for $g(z)$ along some vertical line $\Re z=-\varepsilon$ for sufficiently small $\varepsilon>0$.

We are now ready to perform similar calculations as in Section 2. Thus, we obtain

Theorem 2: The expected maximal lifetime $\mathbb{E}_{n,k}$ of n players each of which has the geometric distribution of order k satisfies

$$\mathbb{E}_{n,k}=\log_{1/a}n-\frac{\gamma+\log A}{\log a}+\frac{1}{2}+\psi(\log_{1/a}n)+O(n^{-\varepsilon})$$

for $0<\varepsilon<\min\left(1,\frac{\log|\alpha_2|}{\log\alpha}\right)$ and a continuous periodic function ψ of period 1 and mean 0 whose Fourier expansion is given by

$$\psi(t)=\frac{1}{\log\alpha}\sum_{x\in\mathbb{Z}\setminus\{0\}}A^{\chi_x}\Gamma(-\chi_x)e^{2x\pi it}=\frac{1}{\log\alpha}\sum_{x\in\mathbb{Z}\setminus\{0\}}\Gamma(-\chi_x)e^{2x\pi i(t-\log_{1/\alpha}A)}$$

where $\chi_x=\frac{2x\pi i}{\log\alpha}$.

By *bootstrapping* we find that, for $k\to\infty$,

$$\alpha\sim 1-qp^k+\cdots \quad\text{and}\quad A\sim 1+kqp^k+\cdots.$$

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