AN ALTERNATIVE PROOF OF A UNIQUE REPRESENTATION THEOREM

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This note describes an alternative approach to the proof in [2] of a representation theorem involving negatively subscripted Pell numbers P_{-n} $(n>0)$, namely,

Theorem: The representation of any integer N as

$$N=\sum_{i=1}^{\infty} a_i P_{-i} \tag{1}$$

where $a_i=0,1,2$ and $a_i=2 \Rightarrow a_{i+1}=0$, is unique and minimal.

To conserve space and avoid unnecessary repetition, we assume that the notation and results in [2] will be familiar to the reader. Our alternative treatment, however, requires the fresh result:

$$2\sum_{i=1}^{n-1}(-1)^{i+1}P_{-i}=-1+(-1)^n(P_{-n}+P_{-n-1}). \tag{2}$$

Repeated use of the recurrence relation for P_{-n} leads to (2). Observe [2] that in (2)

$$q_{-n}=P_{-n}+P_{-n-1} \quad (q_{-1}=-1,\ q_0=1,\ q_1=1). \tag{3}$$

Proof of the Theorem: Suppose there are **two** different representations

$$N=\sum_{i=1}^{h} a_i P_{-i}, \qquad a_h\neq 0,\ a_i=2\Rightarrow a_{i+1}=0 \quad (a_i=0,1,2) \tag{4}$$

and

$$N=\sum_{i=1}^{m} b_i P_{-i}, \qquad b_m\neq 0,\ b_i=2\Rightarrow b_{i+1}=0 \quad (b_i=0,1,2). \tag{5}$$

Case I. Assume $h=m$, so that the Pell numbers in (4) and (5) are the same, but the coefficients a_i, b_i are generally different. Write

$$c_i=a_i-b_i \quad (c_i=0,\pm1,\pm2;\ i=1,2,\ldots,m). \tag{6}$$

Subtract (5) from (4) to derive

$$\sum_{i=1}^{m} c_i P_{-i}=0 \quad \text{by (6)}, \tag{7}$$

that is,

$$c_m P_{-m}+\sum_{i=1}^{m-1} c_i P_{-i}=0, \tag{8}$$

whence, by (2), for a maximum or minimum sum, i.e., $c_i=\pm 2$ $(i=1,2,\ldots,m-1)$,

$$c_m P_{-m}+(-1)^m(P_{-m}+P_{-m-1})=1. \tag{9}$$

[The notation of (3) may be used in (9).] We concentrate on $c_m P_{-m}$ since this term dominates the sums (7)-(9).

m even $(P_{-m} < 0)$: Here (9) gives

$$(c_m + 1)P_{-m} + P_{-m-1} = 1. \tag{9a}$$

Now, in (9a),

(i)	$c_m = 0 \Rightarrow$	$q_{-m} = 1$	by (3)
(ii)	$c_m = 1 \Rightarrow$	$P_{-m+1} = 1$	
(iii)	$c_m = 2 \Rightarrow$	$q_{-m+1} = 1$	by (3)

where in (ii) and (iii) the recurrence relation for Pell numbers [2] has been invoked.

m odd $(P_{-m} > 0)$: Here (9) gives

$$(c_m - 1)P_{-m} - P_{-m-1} = 1. \tag{9b}$$

Next, in (9b),

(iv)	$c_m = 0 \Rightarrow$	$-q_{-m} = 1$	by (3)
(v)	$c_m = 1 \Rightarrow$	$-P_{-m-1} = 1$	
(vi)	$c_m = 2 \Rightarrow$	$P_{-m} - P_{-m-1} = 1.$	

All the equations (i)-(vi) involve contradictions. Of these, perhaps (ii) is the least obvious. Let us therefore examine (ii), which is true for $m = 2$ (even) leading to $c_2 = 1, c_1 = 2$ from (ii) and (8). Now $c_2 = 1 = a_2 - b_2$ implies that $a_2 = 2\ (b_2 = 1)$ or $a_2 = 1\ (b_2 = 0)$, i.e., $a_2 \neq 0$, which contradicts $c_1 = 2 = a_1 - b_1$ since this means that $a_1 = 2\ (b_1 = 0)$ and, hence, $a_1 = 2 \Rightarrow a_2 = 0$ by (1). Thus, (i)-(vi) and, ultimately, (7) are impossible.

Similar reasoning applies when $c_m = -1, -2$. Consequently, the assumption in Case 1 is invalid.

Summary of Case I Results: If $h = m$, then $a_i = b_i$ $(i = 1, \ldots, m)$, i.e., the representations (4) and (5) are identical, so that the representation (4), or (1), is unique.

Case II: Assume $h > m$. Then four subcases exist, depending on the parity of h and m. From [2], with n standing for h and m, in turn,

$$-P_{-n} < N \leq -P_{-n-1} \qquad n \text{ odd} \tag{10}$$

and

$$-P_{-n-1} < N \leq -P_{-n} \qquad n \text{ even.} \tag{11}$$

These restrictions impose a range of values upon N for each integer $n > 0$, for example [2],

$$\begin{array}{ll} n = 1: & 0 \leq N \leq 2 \\ n = 2: & -4 \leq N \leq 2 \\ n = 3: & -4 \leq N \leq 12 \\ n = 4: & -28 \leq N \leq 12 \\ n = 5: & -28 \leq N \leq 70, \end{array} \tag{12}$$

the number of integers [= sums (1)] being 3, 7, 17, 41, 99, in turn, which equal q_2, q_3, q_4, q_5, q_6, respectively.

Results (10) and (11) reveal that each number N, as it occurs for the first time in the ranges (12), is represented uniquely and minimally. For instance,

$$-3 = 1 \cdot P_{-1} + 2 \cdot P_{-2} + 0 \cdot P_{-3} + 0 \cdot P_{-4} + 0 \cdot P_{-5} + \cdots$$

has unique and minimal representation $1 \cdot P_{-1} + 2 \cdot P_{-2}$. We conclude that $h \not> m$. Similarly, $h \not< m$. Therefore, $h = m$, and Case 1 and the Summary are true.

Combining all the preceding discussion, we argue that the validity of the Theorem has been justified.

See [2] for further relevant information and [1] for an analogous treatment of representations involving negatively subscripted Fibonacci numbers.

REFERENCES

1. M. W. Bunder. "Zeckendorf Representations Using Negative Fibonacci Numbers." *The Fibonacci Quarterly* **30.2** (1992):111-15.
2. A. F. Horadam. "Unique Minimal Representation of Integers by Negatively Subscripted Pell Numbers." *The Fibonacci Quarterly* **32.3** (1994):202-06.

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NEW EDITORIAL POLICIES

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1. All articles submitted for publication in The Fibonacci Quarterly will be blind refereed.
2. In place of Assistant Editors, The Fibonacci Quarterly will change to utilization of an Editorial Board.