PARTIAL SUMS FOR SECOND-ORDER RECURRENCE SEQUENCES

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1. BACKGROUND MATERIAL

Motivation for this paper comes from a short article [4] in which some relations between a generalized Fibonacci sequence and the sequence of its partial sums were investigated. An opportunity was clearly provided for a deeper exploration of this theme.

Accordingly, the purpose of this paper is

(a) to extend the relations in [4] to generalized Pell numbers with (i) positive and (ii) negative subscripts, and

(b) as an addendum, to expand the results in [4] to generalized Fibonacci numbers having negative subscripts.

Consider the *generalized Pell sequence* $\{P_n\}$ defined for **all** integers n by

$$P_{n+2} = 2P_{n+1} + P_n \qquad P_1 = a,\ P_2 = b\ \ (P_0 = b - 2a). \tag{1.1}$$

When $a = 1$, $b = 2$, the ordinary Pell sequence $\{p_n\}$ is generated, while when $a = 1$, $b = 3$, we derive the sequence $\{q_n\}$ defined by

$$q_{n+2} = 2q_{n+1} + q_n \qquad q_1 = 1,\ q_2 = 3\ \ (q_0 = 1) \tag{1.2}$$

so that $q_n = \frac{1}{2}Q_n$, the n^{th} *Pell-Lucas number* [2]. Thus, we have the tabulation:

$$\begin{array}{lcccccccccc} n: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ p_n: & 0 & 1 & 2 & 5 & 12 & 29 & 70 & 169 & 408 & \cdots \\ q_n: & 1 & 1 & 3 & 7 & 17 & 41 & 99 & 239 & 577 & \cdots \end{array} \tag{1.3}$$

Observe that the numbers in $\{p_n\}$ are alternately even and odd, while those in $\{q_n\}$ are all odd.

The first few numbers in $\{P_n\}$ and the corresponding sums $S_n = \sum_{i=1}^{n} P_i$ are from (1.1) for $n = 1, 2, \ldots, 10$, therefore:

n	P_n	S_n
1	a	a
2	b	$a + b$
3	$a + 2b$	$2a + 3b$
4	$2a + 5b$	$4a + 8b$
5	$5a + 12b$	$9a + 20b$
6	$12a + 29b$	$21a + 49b$
7	$29a + 70b$	$50a + 119b$
8	$70a + 169b$	$120a + 288b$
9	$169a + 408b$	$289a + 696b$
10	$408a + 985b$	$697a + 1681b$

(1.4)

By standard techniques, e.g., use of (1.1) and induction, it is easy to establish that

$$P_n = ap_{n-2} + bp_{n-1} \quad (n \geq 1,\ p_{-1} = 1 \text{ [see (3.1)]}) \tag{1.5}$$

and

$$S_n = \frac{P_n + P_{n+1} + a - b}{2}, \tag{1.6}$$

whence we deduce the recurrence

$$S_{n+2} = 2S_{n+1} + S_n + b - a \quad (S_0 = 0 \neq P_0 \text{ [see (1.1)]}). \tag{1.7}$$

For subsequent calculations, we will need the *Binet forms*

$$p_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.8}$$

and

$$q_n = \frac{\alpha^n + \beta^n}{2}, \tag{1.9}$$

where

$$\alpha = 1 + \sqrt{2},\ \beta = 1 - \sqrt{2}, \text{ so } \alpha + \beta = 2,\ \alpha\beta = -1,\ \alpha - \beta = 2\sqrt{2}. \tag{1.10}$$

Use of (1.8)-(1.10) produces the *Simson formulas*

$$p_{n+1}p_{n-1} - (p_n)^2 = (-1)^n \tag{1.11}$$

and

$$q_{n+1}q_{n-1} - (q_n)^2 = (-1)^{n+1}2, \tag{1.12}$$

as well as

$$p_n = p_{n-1} + q_{n-1}, \tag{1.13}$$

$$q_n = p_n + p_{n-1}, \tag{1.14}$$

$$\frac{q_n}{p_n} \to \sqrt{2} \text{ as } n \to \infty, \tag{1.15}$$

and the Binet forms for P_n and S_n.

Repeated use of the recurrence relations (1.1) for $\{p_n\}$, where $a = 1, b = 2$, and (1.2) for $\{q_n\}$, where $a = 1, b = 3$, respectively, lead to

$$\sum_{i=1}^{n} p_i = \frac{p_n + p_{n+1} - 1}{2} = \frac{q_{n+1} - 1}{2} \quad \text{by (1.14)} \tag{1.16}$$

and

$$\sum_{i=1}^{n} q_i = p_{n+1} - 1. \tag{1.17}$$

After considerable laborious, but nonetheless satisfying, calculations involving the above equations as appropriate, we determine the Simson formulas for P_n and S_n from (1.5) and (1.6), namely,

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n(a^2 + 2ab - b^2) \tag{1.18}$$

and $(n \geq 1)$

$$S_{n+1}S_{n-1}-S_n^2=\frac{1}{2}\{(-1)^n(a^2+2ab-b^2)+a^2q_{n-2}-b^2q_{n-1}+2abp_{n-2}\}. \tag{1.19}$$

Accordingly, when $n=5$ for instance, $S_6S_4-S_5^2=3a^2+4ab-8b^2$ from (1.19) or directly from (1.4), while $P_6P_4-P_5^2=-a^2-2ab+b^2$ from (1.18) or (1.4). [Who would ever have surmised anything like (1.19)?]

Important special cases of (1.19) arise when $a=1, b=2$ (for p_n), and $a=1, b=3$ (for q_n).

Generally, $S_0=0\neq P_0=b-2a$, unless $b=2a$. Expressed otherwise, P_0 is not part of the summation process.

2. PARTIAL SUMS: POSITIVE SUBSCRIPTS

A basic set of theorems on partial sums can now be established, of which only the first will show the detail.

Theorem 1: $S_{4n}=q_{2n}(aq_{2n-1}+bq_{2n})+a-b$.

Proof:

$$\begin{aligned}
S_{4n}&=\frac{P_{4n}+P_{4n+1}+a-b}{2} && \text{by (1.6)}\\
&=\frac{a(p_{4n-2}+p_{4n-1})+b(p_{4n-1}+p_{4n})+a-b}{2} && \text{by (1.5)}\\
&=\frac{a(\alpha^{4n-1}+\beta^{4n-1}-2)+b(\alpha^{4n}+\beta^{4n}+2)}{4}+a-b && \text{by (1.8), (1.10)}\\
&=aq_{2n}q_{2n-1}+b(q_{2n})^2+a-b && \text{by (1.9)}\\
&=q_{2n}(aq_{2n-1}+bq_{2n})+a-b.
\end{aligned}$$

Likewise,

Theorem 2: $S_{4n-2}=q_{2n-1}(aq_{2n-2}+bq_{2n-1})$.

Theorem 3: $S_{4n+1}=q_{2n}(aq_{2n}+bq_{2n+1})-b$.

Theorem 4: $S_{4n-1}=q_{2n}(aq_{2n-2}+bq_{2n-1})-a$.

Special cases occur when $a=1, b=2$ (i.e., the Pell sequence $\{p_n\}$), namely, for $s_n=\sum_{i=1}^n p_i$,

$$s_{4n}=q_{2n}q_{2n-1}-1, \tag{2.1}$$

$$s_{4n-2}=q_{2n}q_{2n-1}, \tag{2.2}$$

$$s_{4n+1}=(q_{2n+1})^2, \tag{2.3}$$

$$s_{4n-1}=(q_{2n})^2-1=2(p_{2n})^2. \tag{2.4}$$

All four formulas (2.1)-(2.4) may be incorporated into the one neat expression [see (1.16)],

$$s_n=\frac{q_{n+1}-1}{2}\quad (s_0=0), \tag{2.5}$$

[where we have invoked (1.6), ($P_n\equiv p_n$ here), and (1.14)].

However, a virtue of the forms (2.1)-(2.4) is that they display various obvious divisibility properties. Thus, $q_{2n}|s_{4n-2}, q_{2n-1}|s_{4n-2}, p_{2n}|s_{4n-1}$; in particular, $n=3$ in (2.2) gives $4059 = 41.99$, and $n=3$ in (2.4) gives $9800 = 2(70)^2$. As an example of (2.5), $s_8 = 696 = \frac{q_9-1}{2}$ from (1.3).

Observe also the important recurrence from (1.7),

$$s_{n+2} = 2s_{n+1} + s_n + 1 \quad (s_0 = 0). \tag{2.6}$$

Next, write $s'_n = \sum_{i=1}^{n} q_i$. Then $a=1, b=3$ (i.e., the sequence $\{q_n\}$) in (1.6) lead to

$$s'_n = p_{n+1} - 1 \quad (s'_0 = 0), \tag{2.7}$$

i.e. (1.17), since $p_{n+1} = \frac{q_n+q_{n+1}}{2}$ by (1.8) and (1.9), and in (1.7) lead to the recurrence

$$s'_{n+2} = 2s'_{n+1} + s'_n + 2 \quad (s'_0 = 0). \tag{2.8}$$

Let

$$\sigma_n = s'_n - s_n. \tag{2.9}$$

Then, from (2.5) and (2.7), it follows that the sequence $\{\sigma_n\}$ is

$$\begin{array}{rcccccccc} n=1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ s'_n=1 & 4 & 11 & 28 & 69 & 168 & 407 & 984 & \cdots \\ s_n=1 & 3 & 8 & 20 & 49 & 119 & 288 & 696 & \cdots \\ \sigma_n=0 & 1 & 3 & 8 & 20 & 49 & 119 & 288 & \cdots \end{array} \tag{2.10}$$

from which, by (2.6), (2.8), and (2.9), we derive the recurrence [cf. (2.6)]

$$\sigma_{n+2} = 2\sigma_{n+1} + \sigma_n + 1 \quad (\sigma_0 = 0). \tag{2.11}$$

Reverting to (1.4), we notice that

$$S_n = a(\sigma_{n-1} + 1) + b\sigma_n. \tag{2.12}$$

From (1.3) and (2.10),

$$q_n = \sigma_{n+1} - \sigma_{n-1} \tag{2.13}$$

and

$$\sigma_n = \frac{q_n - 1}{2}, \tag{2.14}$$

while, from (1.12), we have the Simson formula for $\{\sigma_n\}$,

$$\sigma_{n+1}\sigma_{n-1} - \sigma_n^2 = \frac{1}{2}\left\{(-1)^{n+1} - q_{n-1}\right\}. \tag{2.15}$$

Other properties of the sequences which flow from the above data include

$$s_n = \sigma_{n+1}, \tag{2.16}$$

$$s'_n = \sigma_n + \sigma_{n+1} = p_{n+1} - 1, \tag{2.17}$$

$$s_n - s_{n-1} = p_n, \tag{2.18}$$

$$s'_n - s'_{n-1} = q_n, \tag{2.19}$$

$$s_n - s_{n-2} = q_n, \tag{2.20}$$

$$s'_n - s'_{n-2} = 2p_n. \tag{2.21}$$

Some of the above features are interrelated, e.g., (2.14) and (2.16) together confirm (2.5).

Observe, from (1.4), (2.10), and (2.12), that σ_n is the coefficient of b in S_n. Another way of arriving at this conclusion is to recall that in (2.9) $a = 1$ for both $\{p_n\}$ and $\{q_n\}$ while $b = 3$ for $\{q_n\}$ but $b = 2$ for $\{p_n\}$, i.e., a "b" difference of $3 - 2 = 1$.

Similar remarks apply later in relation to (1.4a), (2.10a), and (2.12a).

3. PARTIAL SUMS: NEGATIVE SUBSCRIPTS

Corresponding to the results for positive subscripts in the previous section, we have, for negative subscripts,

n:	1	2	3	4	5	6	7	8	$\cdots$	
p_{-n}:	1	−2	5	−12	29	−70	169	−408	$\cdots$	(1.3a)
q_{-n}:	−1	3	−7	17	−41	99	−239	577	$\cdots$	

since

$$p_{-n} = (-1)^{n+1} p_n \tag{3.1}$$

and

$$q_{-n} = (-1)^n q_n, \tag{3.2}$$

as may be readily demonstrated.

Tabulating the simplest expressions in the generalized Pell sequence $\{P_{-n}\}$, and the corresponding sequence of sums $\{S_{-n}\}$ which begins afresh with $S_{-1} = P_{-1}$, gives:

n	P_{-n}			S_{-n}			
1	$5a$	$-$	$2b$	$5a$	$-$	$2b$	
2	$-12a$	$+$	$5b$	$-7a$	$+$	$3b$	
3	$29a$	$-$	$12b$	$22a$	$-$	$9b$	
4	$-70a$	$+$	$29b$	$-48a$	$+$	$20b$	(1.4a)
5	$169a$	$-$	$70b$	$121a$	$-$	$50b$	
6	$-408a$	$+$	$169b$	$-287a$	$+$	$119b$	
7	$985a$	$-$	$408b$	$698a$	$-$	$289b$	
8	$-2378a$	$+$	$985b$	$-1680a$	$+$	$696b$	

Clearly,

$$P_{-n} = ap_{-n-2} + bp_{-n-1} \quad [P_0 = b - 2a \text{ as in (1.1)}]. \tag{1.5a}$$

Write $s_{-n} = \sum_{i=1}^{n} p_{-i}$. Then, as for (1.16), we obtain

$$s_{-n} = \frac{-p_n - p_{-n-1} + 1}{2} = \frac{-q_{-n} + 1}{2} \quad (s_0 = 0) \tag{1.16a}$$

since, by (1.8) and (1.9),

$$q_{-n} = p_{-n} + p_{-n-1}. \tag{1.14a}$$

With a little effort, we derive

$$S_{-n} = \frac{-P_{-n} - P_{-n-1} + 3a - b}{2} = a(s_{-n-2} + 1) + b(s_{-n-1} - 1) \quad (S_0 = 0) \tag{1.6a}$$

and the recurrence

$$S_{-n+2} = 2S_{-n+1} + S_{-n} - 3a + b \quad (S_0 = 0). \tag{1.7a}$$

Paralleling the procedures in the previous section, we have the following four theorems.

Theorem 1a: $S_{-4n} = q_{2n}(-aq_{2n+2} + bq_{2n+1}) + 3a - b.$

Theorem 2a: $S_{-4n+2} = q_{2n}(-aq_{2n} + bq_{2n-1}) + 2a.$

Theorem 3a: $S_{-4n+1} = q_{2n}(aq_{2n+1} - bq_{2n}) + a.$

Theorem 4a: $S_{-4n-1} = q_{2n+1}(aq_{2n+2} - bq_{2n+1}) + 2a - b.$

Putting $a = 1, b = 2$, we have the Pell numbers results:

$$s_{-4n} = -(q_{2n})^2 + 1, \tag{2.1a}$$

$$s_{-4n+2} = -(q_{2n-1})^2, \tag{2.2a}$$

$$s_{-4n+1} = q_{2n}q_{2n-1} + 1, \tag{2.3a}$$

$$s_{-4n-1} = q_{2n}q_{2n+1}. \tag{2.4a}$$

Fortunately, (2.1a)-(2.4a) may be amalgamated into one pleasing form [cf. (1.16a)],

$$s_{-n} = \frac{-q_n + 1}{2}. \tag{2.5a}$$

Furthermore, from (1.7a),

$$s_{-n+2} = 2s_{-n+1} + s_{-n} - 1. \tag{2.6a}$$

Coming now to the special case $a = 1, b = 3$ again, we see that, denoting $s'_{-n} = \sum_{i=1}^{n} q_{-i}$,

$$s'_{-n} = -p_{-n} \quad (s'_0 = 0) \tag{2.7a}$$

and

$$s'_{-n+2} = 2s'_{-n+1} + s'_{-n}. \tag{2.8a}$$

Writing

$$\sigma_{-n} = s'_{-n} - s_{-n}, \tag{2.9a}$$

we may tabulate values of $\{\sigma_{-n}\}$ as in (2.10) with a recurrence corresponding to (2.11), thus,

$$\begin{array}{rrrrrrrrrl} n = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ s'_{-n} = & -1 & 2 & -5 & 12 & -29 & 70 & -169 & 408 & \cdots \\ s_{-n} = & 1 & -1 & 4 & -8 & 21 & -49 & 120 & -288 & \cdots \\ \sigma_{-n} = & -2 & 3 & -9 & 20 & -50 & 119 & -289 & 696 & \cdots \end{array} \tag{2.10a}$$

whence

$$\sigma_{-n+2} = 2\sigma_{-n+1} + \sigma_{-n} + 1 \quad (\sigma_0 = 0). \tag{2.11a}$$

It follows from (1.4a) that

$$S_{-n} = a(\sigma_{-n-1} + 2) + b\sigma_{-n}. \tag{2.12a}$$

Furthermore,

$$q_{-n} = \sigma_{-n} - \sigma_{-n+2} \tag{2.13a}$$

while

$$\sigma_{-n} = \frac{-q_{-n-1} - 1}{2}. \tag{2.14a}$$

Additional results include

$$s_{-n} = \sigma_{-n+1} + 1, \tag{2.16a}$$

$$s'_{-n} = -\sigma_{-n+1} + \sigma_{-n+2}, \tag{2.17a}$$

$$s_{-n-1} - s_{-n} = p_{-n-1}, \tag{2.18a}$$

$$s'_{-n-1} - s'_{-n} = q_{-n-1}, \tag{2.19a}$$

$$s_{-n-2} - s_{-n} = q_{-n-1}, \tag{2.20a}$$

$$s'_{-n-2} - s'_{-n} = 2p_{-n-1}. \tag{2.21a}$$

One may also ascertain that

$$\begin{cases} \sigma_n + \sigma_{-n+1} = -1 & n \text{ even}, \\ \sigma_n - \sigma_{-n+1} = 0 & n \text{ odd}. \end{cases} \tag{3.3}$$

Properties of $\{\sigma_n\}$ are the subject of another paper, so we do not pursue the occurence of it in this exposition.

Other facets of the patterns in $P_{\pm n}$ and $S_{\pm n}$ may be recorded:

$$P_n + (-1)^{n-1} P_{-n+2} = 2aq_{n-1}, \tag{3.4}$$

$$P_n + (-1)^n P_{-n+2} = 2(-a+b)p_{n-1}, \tag{3.5}$$

$$P_n + (-1)^n P_{-n+4} = 2bq_{n-2}, \tag{3.6}$$

$$P_n + (-1)^{n-1} P_{-n+4} = 2(a+b)p_{n-2}, \tag{3.7}$$

$$S_{2n} + S_{-2n+1} = 2ap_{2n} + 2a - b, \tag{3.8}$$

$$S_{2n} - S_{-2n+1} = (-a+b)q_{2n} - a, \tag{3.9}$$

$$S_{2n+1} + S_{-2n} = (-a+b)q_{2n+1} + 2a - b, \tag{3.10}$$

$$S_{2n+1} - S_{-2n} = 2ap_{2n+1} - a. \tag{3.11}$$

Simson formulas for P_{-n} and S_{-n} may be obtained in the manner used for (1.18) and (1.19). In the first instance,

$$P_{-n-1}P_{-n+1} - P_{-n}^2 = (-1)^n (a^2 + 2ab - b^2), \tag{1.18a}$$

i.e., (1.18) is valid for all n. Discovery of the negative-subscript Simson analogue of (1.19) (with specializations for s_{-n} and s'_{-n}) is left to the spirit of enquiry and adventure of the reader (to be attempted because it is **there**!).

4. THE FIBONACCI CASE

A more expansive treatment of [4] will now be outlined. Ordinary Fibonacci and Lucas numbers will be represented by f_n and ℓ_n, respectively, while the upper-case notation F_n for the *generalized Fibonacci number* will be retained. To avoid confusion, we will use $T_n = \sum_{i=1}^n F_i$ Basic properties of $\{f_n\}$ and $\{\ell_n\}$ will be assumed.

Mutatis mutandis, we have [4]

$$T_n = F_{n+2} - b = af_n + b(f_{n+1} - 1) \quad (T_0 = 0 \neq F_0 = -a + b) \tag{4.1}$$

with, in particular,

$$T_{4n} = \ell_{2n}F_{2n+2} - 2b \quad [= F_{4n+2} - b \text{ from } (4.1)] \tag{4.2}$$

and

$$T_{4n-2} = \ell_{2n-1}F_{2n+1} \quad [= F_{4n} - b \text{ from } (4.1)]. \tag{4.3}$$

Moreover [4], there is the recurrence

$$T_{n+2} = T_{n+1} + T_n + b. \tag{4.4}$$

If $a = 1, b = 1$, and if we write $t_n = \Sigma_{i=1}^n f_i$, then

$$t_n = f_{n+2} - 1, \tag{4.5}$$

$$t_{4n} = \ell_{2n}f_{2n+2} - 2 \quad [= f_{4n+2} - 1 \text{ from } (4.5)], \tag{4.6}$$

and

$$t_{4n-2} = \ell_{2n-1}f_{2n+1} \quad [= f_{4n} - 1 \text{ from } (4.5)], \tag{4.7}$$

so that $\ell_{2n-1} | t_{4n-2}$, $f_{2n+1} | t_{4n-2}$, e.g., for $n = 4$, $(\ell_7 = 29) | 986$ and $(f_9 = 34) | 986$. Furthermore, (4.4) yields the recurrence

$$t_{n+2} = t_{n+1} + t_n + 1 \quad (t_0 = 0). \tag{4.8}$$

Instead of focusing on f_n, suppose we put $a = 1, b = 3$ and write $t'_n = \Sigma_{i=1}^n \ell_i$. Then

$$t'_n = \ell_{n+2} - 3, \tag{4.9}$$

$$t'_{4n} = \ell_{2n}\ell_{2n+2} - 6 \quad [= \ell_{4n+2} - 3 \text{ from } (4.9)], \tag{4.10}$$

and

$$t'_{4n-2} = \ell_{2n-1}\ell_{2n+1} \quad [= \ell_{4n} - 3 \text{ from } (4.9)], \tag{4.11}$$

with the recurrence

$$t'_{n+2} = t'_{n+1} + t_n + 3 \quad (t'_0 = 0). \tag{4.12}$$

Again, observe the factorization and divisibility in (4.11).

Table 1 lists values of T_n, t_n, t'_n [and τ_n (4.15)].

TABLE 1. Partial Sums for F_n ($n = 1, 2, \ldots, 10$)

n	F_n	T_n	t_n	t'_n	τ_n
1	a	a	1	1	0
2	b	$a + b$	2	4	2
3	$a + b$	$2a + 2b$	4	8	4
4	$a + 2b$	$3a + 4b$	7	15	8
5	$2a + 3b$	$5a + 7b$	12	26	14
6	$3a + 5b$	$8a + 12b$	20	44	24
7	$5a + 8b$	$13a + 20b$	33	73	40
8	$8a + 13b$	$21a + 33b$	54	120	66
9	$13a + 21b$	$34a + 54b$	88	196	108
10	$21a + 34b$	$55a + 88b$	143	319	176

Negative subscripts are utilized to obtain results paralleling those above. First, however, we remark that [cf. (3.1), (3.2)]

$$f_{-n} = (-1)^{n+1} f_n \tag{4.13}$$

and

$$\ell_{-n} = (-1)^n \ell_n. \tag{4.14}$$

Readers are urged to construct appropriate tables of values for f_{-n} and ℓ_{-n} from (4.13) and (4.14). See Table 2 for $T_{-n} = \sum_{i=1}^{n} F_{-i}$ and hence for t_{-n} and t'_{-n} [and τ_{-n} (4.15a)].

TABLE 2. Partial Sums for F_{-n} ($n = 1, 2, \ldots, 10$)

n	F_{-n}	T_{-n}	t_{-n}	t'_{-n}	τ_{-n}
1	$2a - b$	$2a - b$	1	−1	−2
2	$-3a + 2b$	$-a + b$	0	2	2
3	$5a - 3b$	$4a - 2b$	2	−2	−4
4	$-8a + 5b$	$-4a + 3b$	−1	5	6
5	$13a - 8b$	$9a - 5b$	4	−6	−10
6	$-21a + 13b$	$-12a + 8b$	−4	12	16
7	$34a - 21b$	$22a - 13b$	9	−17	−26
8	$-55a + 34b$	$-33a + 21b$	−12	30	42
9	$89a - 55b$	$56a - 34b$	22	−46	−68
10	$-144a + 89b$	$-88a + 55b$	−33	77	110

Repeated application of the recurrence relation for $\{F_n\}$, with the initial conditions, yields

$$T_{-n} = -F_{-n+1} + a \quad (T_0 = 0). \tag{4.1a}$$

In particular,

$$T_{-10} = -F_{-9} + a = -88a + 55b = 11(-8a + 5b) = \ell_5 F_{-4} \quad (\text{i.e., } \ell_5 | T_{-10}, F_{-4} | T_{-10}).$$

Accordingly,

$$t_{-n} = -f_{-n+1} + 1 \quad (t_0 = 0), \tag{4.5a}$$

and

$$t'_{-n} = -\ell_{-n+1} + 1 \quad (t'_0 = 0). \tag{4.9a}$$

Setting

$$\tau_n = t'_n - t_n \quad (\tau_0 = 0), \tag{4.15}$$

we discover [cf. (2.11)-(2.15)] the following:

$$\tau_n = 2(f_{n+1} - 1) = 2t_{n-1}, \tag{4.16}$$

$$\tau_n - \tau_{n-2} = 2f_n, \tag{4.17}$$

$$\tau_{n+2} = \tau_{n+1} + \tau_n + 2, \tag{4.18}$$

$$\tau_{n+1}\tau_{n-1} - \tau_n^2 = 4\{(-1)^{n+1} - f_{n-2}\}. \tag{4.19}$$

Moreover,

$$T_n = a\left(\frac{\tau_{n-1}}{2} + 1\right) + b\frac{\tau_n}{2}. \tag{4.20}$$

Replacing n by $-n$ in (4.15) so that

$$\tau_{-n} = t'_{-n} - t_{-n} \quad (\tau_0 = 0), \tag{4.15a}$$

one may obtain a table of values of the numbers in the sequence $\{\tau_{-n}\}$, whence

$$\tau_{-n} = 2(-f_{-n}) = 2(t_{-n} - 1), \tag{4.16a}$$

$$\tau_{-n} - \tau_{-n+2} = 2f_{-n+1} \quad (n \geq 2), \tag{4.17a}$$

$$\tau_{-n+2} = \tau_{-n+1} + \tau_{-n}, \tag{4.18a}$$

$$\tau_{-n-1}\tau_{-n+1} - \tau_{-n}^2 = 4(-1)^n, \tag{4.19a}$$

$$T_{-n} = a\frac{\tau_{-n-2}}{2} + b\frac{\tau_{-n}}{2}. \tag{4.20a}$$

Note that $\frac{1}{2}\tau_n$ and $\frac{1}{2}\tau_{-n}$ in (4.20) and (4.20a) are the coefficients of b in T_n and T_{-n}, respectively. Also refer to Tables 1 and 2. The reason for this is that $a = 1$ for both $\{f_n\}$ and $\{\ell_n\}$, but $b = 3$ for $\{\ell_n\}$ and $b = 1$ for $\{f_n\}$, i.e., there is a "b" difference of $3 - 1 = 2$.

Going back now to $\{F_n\}$ and $\{T_n\}$, we discover [cf. (3.4)-(3.11)]:

$$F_n + (-1)^{n+1}F_{-n+2} = a\ell_{n-1} \quad (n \geq 2), \tag{4.21}$$

$$F_n + (-1)^n F_{-n+2} = (-1)^n(-a + 2b)f_{n-1} \quad (n \geq 2), \tag{4.22}$$

$$F_n + (-1)^{n-1}F_{-n+4} = (2a + b)f_{n-2} \quad (n \geq 4), \tag{4.23}$$

$$F_n + (-1)^n F_{-n+4} = b\ell_{n-2}, \tag{4.24}$$

$$T_{2n} + T_{-2n+1} = (2a + b)f_{2n} + a - b, \tag{4.25}$$

$$T_{2n} - T_{-2n+1} = b\ell_{2n} - (a + b), \tag{4.26}$$

$$T_{2n+1} + T_{-2n} = b\ell_{2n+1} + a - b, \tag{4.27}$$

$$T_{2n+1} - T_{-2n} = (2a + b)f_{2n+1} - (a + b). \tag{4.28}$$

No doubt further identities of this *genre* are discoverable.

Frequent comparison of corresponding outcomes for the Pell and Fibonacci cases is both desirable and instructive. In this context, discovery of the Simson formulas for F_n, T_n, t_n, and t_n' (for $n > 0, n < 0$)—some of them not a pretty sight!—might be undertaken.

The Additions $s_n + s_n'$ and $t_n + t_n'$

Instead of considering the differences $\sigma_n = s_n' - s_n$ and $\tau_n = t_n' - t_n$, suppose the additions $\kappa_n = s_n' + s_n$ and $\lambda_n = t_n' + t_n$ are examined.

Consider then Table 3,

TABLE 3. Addition of Partial Sums

	$n =$	1	2	3	4	5	6	7	8	$\cdots$
κ_n	$= s_n' + s_n =$	2	7	19	48	118	287	695	1680	$\cdots$
κ_{-n}	$= s_{-n}' + s_{-n} =$	0	1	−1	4	−8	21	−49	120	$\cdots$
λ_n	$= t_n' + t_n =$	2	6	12	22	38	64	106	174	$\cdots$
λ_{-n}	$= t_{-n}' + t_{-n} =$	0	2	0	4	−2	8	−8	18	$\cdots$

in which

$$\kappa_0 = 0, \tag{4.29}$$

$$\lambda_0 = 0, \tag{4.30}$$

whence

$$\kappa_n = \sigma_{n+2} - 1 = s_{n+1} - 1, \tag{4.31}$$

$$\kappa_{n+2} = 2\kappa_{n+1} + \kappa_n + 3, \tag{4.32}$$

$$\kappa_{n+2} - \kappa_n = q_{n+3}, \tag{4.33}$$

$$\kappa_{n+1} - \kappa_n = p_{n+1}. \tag{4.34}$$

Moreover,

$$\kappa_{-n} = \sigma_{-n+2} + 1, \tag{4.31a}$$

$$\kappa_{-n+2} = 2\kappa_{-n+1} + \kappa_{-n} - 1. \tag{4.32a}$$

On the other hand,

$$\lambda_n = \tau_{n+2} - 2, \tag{4.35}$$

$$\lambda_{n+2} = \lambda_{n+1} + \lambda_n + 4, \tag{4.36}$$

$$\lambda_{n+2} - \lambda_n = 2f_{n+4}, \tag{4.37}$$

$$\lambda_{n+1} - \lambda_n = 2f_{n+2}, \tag{4.38}$$

while

$$\lambda_{-n} = \tau_{-n+2} + 2, \tag{4.35a}$$

$$\lambda_{-n+2} = \lambda_{-n+1} + \lambda_{-n} - 2. \tag{4.36a}$$

Aware of the opportunities offered by this amplification of our theory, we may develop properties corresponding to those for differences until satiated.

5. CONCLUDING REMARKS

Finally, there are a few thoughts worthy of further consideration.

(a) Other pairs of sequences related like $\{f_n\}$ and $\{\ell_n\}$, and $\{p_n\}$ and $\{q_n\}$ exist. Our results above suggest analogous—if, perhaps, less interesting—properties for such pairs.

(b) Sequences $\{\sigma_n\}$ and $\{\frac{1}{2}\tau_n\}$ $(n > 0)$ occur naturally *inter alia* in the minimal and maximal representations of positive integers by Pell and Fibonacci numbers, respectively. The former sequence is part of the stimulus for a separate research program.

(c) Recurrences of the form

$$R_{n+2} = kR_{n+1} + R_n + c \quad (k, c \text{ constants}) \tag{5.1}$$

appear in many guises in this paper, for example, when $R_n = S_n, s_n, s'_n, \sigma_n, \kappa_n, T_n, t_n, t'_n, \tau_n$, and λ_n, with extensions to negative subscripts. Such recurrences (5.1) arise in other circum stances, e.g., in a graph-theoretic context, and are the subject of a separate investigation.

(d) Numbers q_n of the sequence $\{q_n = \frac{1}{2}Q_n\}$, where Q_n are the *Pell-Lucas numbers*, feature prominently in a variety of papers. They (and p_n) have been called the *Eudoxus numbers* [1], though their first "official" appearance, according to [5], seems to have been in [3] in 1916, while some of the properties of q_n in relation to p_n have been recorded in [6] in 1949.

Can anyone tell me if there is any justification for the name "Eudoxus numbers" to describe the members of these interesting sequences? After all, the life-span of the ancient Greek mathematical genius, Eudoxus (ca. 408-355 B.C.), is a very far off human event.

Many, indeed, have been the fascinating and pleasurable ramifications of our modest attempt to expand the brief material in [4]. Evidently, there is much scope for further exploration and discovery in this field. Mindful of our stated objectives, however, we rest our case at this point.

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