# A NOTE ON BROWN AND SHIUE'S PAPER ON A REMARK RELATED TO THE FROBENIUS PROBLEM


**Öystein J. Rödseth**

Department of Mathematics, University of Bergen, Allégt. 55, N-5007 Bergen, Norway



(*Submitted April 1993*)


Given relatively prime positive integers $a$, $b$, let NR denote the set of positive integers with no representation by the linear form $ax + by$ in nonnegative integers $x, y$. It is well known that the set NR is finite. For a nonnegative integer $m$, we put

$$S_m(a,b) = \sum_{n \in \mathrm{NR}} n^m.$$

Sylvester [3] showed that $\#\mathrm{NR} = S_0(a,b) = \frac{1}{2}(a-1)(b-1)$ and, recently, Brown and Shiue [1] found a similar closed form for $S_1(a,b)$. Brown and Shiue did this by determining a closed form for the generating function $f(x)$ of the characteristic function of the set NR and then computing $f'(1) = S_1(a,b)$. In this note we use a more direct approach, which gives us a closed form for $S_m(a,b)$ valid for every nonnegative integer $m$.

Let integers $n$, $r$, $s$ be connected by the relations

$$r \equiv n \pmod{a},\ 0 \le r < a; \quad bs \equiv r \pmod{a},\ 0 \le s < a.$$

We have that $n \in \mathrm{NR}$ if and only if $n = -at + bs$ for some integer $t$ in the interval $1 \le t \le \lfloor bs/a \rfloor$, that is, if and only if $n = ak + r$ for some integer $k$ in the interval $0 \le k \le (bs-r)/a - 1$. Hence,

$$S_m(a,b) = \sum_{r=0}^{a-1} \sum_{k=0}^{\frac{bs-r}{a}-1} (ak+r)^m.$$

For the exponential generating function of the sequence $\{S_m\}$, this gives

$$\begin{aligned}\sum_{m=0}^{\infty} S_m(a,b)\frac{z^m}{m!} &= \sum_{r=0}^{a-1} \sum_{k=0}^{\frac{bs-r}{a}-1} \sum_{m=0}^{\infty} (ak+r)^m \frac{z^m}{m!} \\ &= \sum_{r=0}^{a-1} \sum_{k=0}^{\frac{bs-r}{a}-1} e^{(ak+r)z} = \frac{1}{e^{az}-1}\left(\sum_{r=0}^{a-1} e^{bsz} - \sum_{r=0}^{a-1} e^{rz}\right).\end{aligned}$$

As $r$ runs through the set $\{0, 1, \ldots, a-1\}$, so does $s$. Hence,

$$\sum_{r=0}^{a-1} e^{bsz} = \sum_{s=0}^{a-1} e^{bsz},$$

and we find that

$$\sum_{m=0}^{\infty} S_m(a,b)\frac{z^m}{m!} = \frac{e^{abz}-1}{(e^{az}-1)(e^{bz}-1)} - \frac{1}{e^z-1}.$$

Multiplying this relation by $z$ gives

$$\sum_{m=1}^{\infty} mS_{m-1}(a,b)\frac{z^m}{m!} = \sum_{i=0}^{\infty} B_i a^i \frac{z^i}{i!} \sum_{j=0}^{\infty} B_j b^j \frac{z^j}{j!} \sum_{k=0}^{\infty} \frac{a^k b^k}{k+1} \cdot \frac{z^k}{k!} - \sum_{m=0}^{\infty} B_m \frac{z^m}{m!},$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}, \ldots$ are the Bernoulli numbers; cf. formula (6.81) and section 7.6 in [2]. Equating coefficients of $z^m$ now gives the

***Theorem:*** For $m = 1, 2, \ldots$, we have

$$S_{m-1}(a,b) = \frac{1}{m(m+1)} \sum_{i=0}^{m} \sum_{j=0}^{m-i} \binom{m+1}{i} \binom{m+1-i}{j} B_i B_j a^{m-j} b^{m-i} - \frac{1}{m} B_m.$$

It is not difficult to see that, considered as a polynomial in $a$ and $b$, $S_m(a,b)$ has the algebraic factor $(a-1)(b-1)$. In addition, if $m$ is even $\geq 2$, then $S_m(a,b)$ also has the factor $ab(ab-a-b)$.

Our theorem gives us, of course, Sylvester's result for $S_0$ and Brown and Shiue's formula [1],

$$S_1(a,b) = \frac{1}{12}(a-1)(b-1)(2ab-a-b-1).$$

Also, for $S_2$, we obtain a rather simple formula:

$$S_2(a,b) = \frac{1}{12}(a-1)(b-1)ab(ab-a-b).$$

## REFERENCES

1. T. C. Brown & P. J.-S. Shiue. "A Remark Related to the Frobenius Problem." *The Fibonacci Quarterly* **31.1** (1993):32-36.
2. R. L. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics.* Reading, Mass.: Addison-Wesley, 1990.
3. J. J. Sylvester. "Mathematical Questions with Their Solutions." *Educational Times* **41** (1884):21.

AMS Classification Numbers: 11B57, 11B68