THE ZECKENDORF ARRAY EQUALS THE WYTHOFF ARRAY

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1. INTRODUCTION

It is well known that every *n* in the set *N* of positive integers is uniquely a sum of nonconsecutive Fibonacci numbers. This sum *n* is known as the Zeckendorf representation of *n*. We arrange these representations to form the Zeckendorf array Z = Z(i, j) as follows: column *j* of *Z* is the increasing sequence of all *n* in whose Zeckendorf representation the least term is F_{j+1} . The first row of *Z* therefore consists of Fibonacci numbers:

$$z(1, 1) = 1 = F_2$$
 $z(1, 2) = 2 = F_3$ $z(1, 3) = 3 = F_4$ \cdots $z(1, j) = F_{i+1} \cdots$

and the second row begins with the numbers 4 = 3 + 1, 7 = 5 + 2, 11 = 8 + 3, and 18 = 13 + 5. The reader is urged to write down several terms of the next two rows before reading further.

The Wythoff array, W = W(i, j), partly shown in Table 1, was introduced by David R. Morrison [9], in connection with Wythoff pairs, which are the winning pairs of numbers in Wythoff's game. (See, for example, [2], [12]). Morrison proved several interesting things about W: every positive integer n occurs exactly once in W, as does every Wythoff pair; every row is a (generalized) Fibonacci sequence [i.e., w(i, j) = w(i, j-1) + w(i, j-2) for every $i \ge 1$ and $j \ge 3$]. In fact, Morrison proved that, in a sense, every positive Fibonacci sequence of integers is a row of W.

TABLE 1. The Wythoff Array

1	2	3	5	8	13	21	34	55	89	144	
4	7	11	18	29	47	76	123	199	322	521	•••
6	10	16	26	42	68	110	178	288	466	754	
9	15	24	39	63	102	165	267	432	699	1131	
12	20	32	52	84	136	220	356	576	932	1508	
14	23	37	60	97	157	254	411	665	1076	1741	
17	28	45	73	118	191	309	500	809	1309	2118	
19	31	50	81	131	212	343	555	898	1453	2351	
22	36	58	94	152	246	398	644	1042	1686	2728	
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Morrison also proved that the first column of W is given by $w(i, 1) = [i\alpha] + i - 1$, where $\alpha = (1 + \sqrt{5})/2$. The rest of W is then given inductively by

$$w(i, j+1) = \begin{cases} [\alpha w(i, j)] + 1 & \text{if } j \text{ is odd,} \\ [\alpha w(i, j)] & \text{if } j \text{ is even,} \end{cases} \text{ for } i = 1, 2, 3, \dots .$$
(1)

2. SHIFTING SUBSCRIPTS: $F_{n+1} \rightarrow F_{n+2}$

We define a shift function $f: N \rightarrow N$ in terms of Zeckendorf representations:

if
$$n = \sum_{h=1}^{\infty} c_h F_{h+1}$$
, then $f(n) = \sum_{h=1}^{\infty} c_h F_{h+2}$.

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Lemma 1: The shift function *f* is a strictly increasing function.

We shall prove Lemma 1 in a more general form in Section 3.

Theorem 1: The first column of the Zeckendorf array Z determines all of Z by the recurrences

$$z(i, j+1) = f(z(i, j))$$
(2)

for all $i \ge 1$ and $j \ge 1$.

Proof: We have $z(1, j) = F_{j+1}$ for all $j \ge 1$, so that row 1 of Z is determined by z(1, 1) = 1 and f. Assume $k \ge 1$ and that (2) holds for all $j \ge 1$, for all $i \le k$. Write the Zeckendorf representation of z(k+1, 1) as $z(k+1, 1) = \sum_{h=1}^{\infty} c_h F_{h+1}$, noting that the following conditions hold:

- (i) $c_1 = 1;$
- (ii) $c_h \in \{0, 1\}$ for every h in N;
- (iii) for every h in N, if $c_h = 1$ then $c_{h+1} = 0$;
- (iv) there exists *n* in *N* such that $c_h = 0$ for every $h \ge n$.

Let m = f(z(k+1, 1)). Then the representation $\sum_{h=1}^{\infty} c'_h F_{h+1}$, where $c'_1 = 0$ and $c'_h = c_{h-1}$ for all $h \ge 2$, is the Zeckendorf representation of m. Also, m is in column 2 of Z, since $c'_1 = 0$ and $c'_2 = 1$. By the induction hypothesis, z(i, 2) = f(z(i, 1)) for i = 1, 2, ..., k, and since column 2 is an increasing sequence, m must lie in a row numbered $\ge k+1$ by Lemma 1. We shall show that this row number cannot be > k+1.

Suppose m > z(k + 1, 2) and let the Zeckendorf representation for z(k + 1, 2) be $\sum_{h=1}^{\infty} d'_h F_{h+1}$. Then the number $q = \sum_{h=1}^{\infty} d_h F_{h+1}$, where $d_h = d'_{h+1}$ for $h \ge 1$, is the Zeckendorf representation for a number having $d_1 = 1$, so that this number lies in column 1 of Z. It is not one of the first k terms, and it is not z(k + 1, 1) since its sequel in row k + 1 is not m. Therefore, q = z(K, 1) for some $K \ge k+2$. We now have z(k+1, 1) < q and f(q) < f(z(k+1, 1)), a contradiction to Lemma 1. Therefore, z(k+1, 2) = f(z(k+1, 1)).

Let $j \ge 2$ and suppose that z(k+1, j) = f(z(k+1, j-1)). The argument just used for j = 2 applies here in the same way, giving z(k+1, j+1) = f(z(k+1, j)). The induction on j is finished, so that (2) holds for all $j \ge 1$ for i = k+1. Consequently, the induction on k is finished, so that (2) holds throughout Z. \Box

Lemma 2:

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 $f(n) = \begin{cases} [\alpha n] + 1 & \text{if } n \text{ is in an odd numbered column of } Z, \\ [\alpha n] & \text{if } n \text{ is in an even numbered column of } Z. \end{cases}$

Proof: The fact that the continued fraction for α is [1, 1, 1, ...] leads as in [10, p. 10] to the well-known inequality

$$\frac{1}{F_{h+2}} < \left| \alpha F_h - F_{h+1} \right| < \frac{1}{F_{h+1}}$$

for h = 1, 2, 3, ..., and this in turn implies

$$\frac{1}{F_{h+2}} < \{\alpha F_h\} < \frac{1}{F_{h+1}} \text{ for odd } h$$
(3)

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and

$$-\frac{1}{F_{h+1}} < \{\alpha F_h\} - 1 < -\frac{1}{F_{h+2}} \text{ for even } h.$$
(4)

Write the Zeckendorf representation of n as indicated by the sum

.

$$n = c_1 F_2 + c_2 F_3 + c_3 F_4 + \cdots$$
 (5)

Then

$$f(n) = c_1 F_3 + c_2 F_4 + c_3 F_5 + \cdots$$
(6)

Also,

$$n\alpha = c_1 \alpha F_2 + c_2 \alpha F_3 + c_3 \alpha F_4 + \cdots$$

= $c_1 (F_3 + \{\alpha F_2\} - 1) + c_2 (F_4 + \{\alpha F_3\}) + c_3 (F_5 + \{\alpha F_4\} - 1) + \cdots$
= $f(n) + \mathcal{G}_1(n) + \mathcal{G}_2(n),$

where

$$\mathcal{G}_1(n) = c_1(\{\alpha F_2\} - 1) + c_3(\{\alpha F_4\} - 1) + c_5(\{\alpha F_6\} - 1) + \cdots$$

and

$$\mathcal{G}_{2}(n) = c_{2}\{\alpha F_{3}\} + c_{4}\{\alpha F_{5}\} + c_{6}\{\alpha F_{7}\} + \cdots$$

<u>Case 1</u>: *n* is an even numbered column of Z. In this case, the least nonzero coefficient c_H in (5) has an even index H, so that

$$\mathcal{G}_{1}(n) + \mathcal{G}_{2}(n) = \{\alpha F_{H+1}\} + \text{other terms}$$

$$\leq \{\alpha F_{H+1}\} + \{\alpha F_{H+3}\} + \dots < \frac{1}{F_{H+2}} + \frac{1}{F_{H+4}} + \frac{1}{F_{H+6}} + \dots \text{ by } (3)$$

$$\leq \frac{1}{2} + \frac{1}{5} + \frac{1}{13} + \dots < \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1.$$

Also,

$$\mathcal{G}_{1}(n) + \mathcal{G}_{2}(n) \geq \{\alpha F_{H+1}\} + (-1 + \{\alpha F_{H+4}\}) + (-1 + \{\alpha F_{H+6}\}) + \cdots$$

$$\geq \frac{1}{F_{H+3}} - \frac{1}{F_{H+5}} - \frac{1}{F_{H+7}} - \cdots \text{ by (3) and (4)}$$

$$\geq \frac{1}{F_{H+3}} - \frac{1}{F_{H+5}} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) \geq \frac{1}{F_{H+3}} - \frac{2}{F_{H+5}} > 0.$$

The conclusion in Case 1 is that $f(n) = [n\alpha]$.

<u>Case 2</u>: *n* is an odd numbered column of *Z*. Then the least nonzero coefficient c_H in (5) has an odd index *H*, and

$$\mathcal{G}_{1}(n) + \mathcal{G}_{2}(n) = -1 + \{\alpha F_{H+1}\} + \text{other terms}$$

$$\leq -1 + \{\alpha F_{H+1}\} + \{\alpha F_{H+4}\} + \{\alpha F_{H+6}\} + \cdots$$

$$< -\frac{1}{F_{H+3}} + \frac{1}{F_{H+5}} + \frac{1}{F_{H+7}} + \cdots \text{ by (3) and (4)}$$

$$\leq -\frac{1}{F_{H+3}} + \frac{2}{F_{H+5}} < 0.$$

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$$\begin{aligned} \mathcal{G}_{1}(n) + \mathcal{G}_{2}(n) &\geq -1 + \{\alpha F_{H+1}\} + (-1 + \{\alpha F_{H+3}\}) + (-1 + \{\alpha F_{H+5}\}) + \\ &> -\frac{1}{F_{H+2}} - \frac{1}{F_{H+4}} - \frac{1}{F_{H+6}} - \cdots \text{ by (4)} \\ &> -\frac{2}{F_{H+2}} \geq -1. \end{aligned}$$

The conclusion in Case 2 is that $f(n) = [n\alpha] + 1$. \Box

Theorem 2: The Zeckendorf array equals the Wythoff array.

Proof: Let C be the set of numbers in the first column of Z. Let S be the complement of C in the set of positive integers. Let $\{s_n\}$ be the sequence obtained by arranging the elements of S in increasing order. It is known [4] that this sequence of one-free Zeckendorf sums is given by $s_n = [(n+1)\alpha] - 1$, n = 1, 2, 3, ... We shall apply Beatty's theorem (see [1], [11]) on complementary sequences of positive integers to prove that $z(i, 1) = [i\alpha] + i - 1$: first, $\frac{1}{\alpha} + \frac{1}{\alpha+1} = 1$, so that, by Beatty's theorem, the sequences $[n\alpha]$ and $[i\alpha] + i$ are complementary; this implies that the sets $\{[(n+1)\alpha]\}$ and $\{[i\alpha]+i\} \cup \{1\}$ partition N, which in turn implies that the sequences s_n and z(i, 1) are complementary. Since $w(i, 1) = [i\alpha] + i - 1$, we have z(i, 1) = w(i, 1). Now the recurrence (1) together with Theorem 1 and Lemma 2 imply that Z = W.

3. HIGHER-ORDER ZECKENDORF ARRAYS

Let *m* be an integer ≥ 2 . Define a sequence $\{s_i\}$ inductively, as follows:

 $s_i = 1$ for i = 1, 2, 3, ..., m, $s_i = s_{i-1} + s_{i-m}$ for i = m+1, m+2, ...,

and define the Zeckendorf *m*-basis as the sequence $\{b_j^{(m)}\}$, where $b_j^{(m)} = s_{m+j-1}$ for all *j* in *N*. It is proved in [5] and probably elsewhere that every *n* in *N* is uniquely a sum

$$b_{i_1}^{(m)} + b_{i_2}^{(m)} + \dots + b_{i_v}^{(m)}$$
, where $i_t - i_u \ge m$ whenever $t > u$. (7)

We call the sum in (7) the *m*-order Zeckendorf representation of *n*, and we define the *m*-order Zeckendorf array $Z^{(m)} = Z^{(m)}(i, j)$ as follows: column *j* of $Z^{(m)}$ is the increasing sequence of all *n* in whose *m*-order Zeckendorf representation the least term is $b_j^{(m)}$. The first row of $Z^{(m)}$ is the Zeckendorf *m*-basis. Of course, the Zeckendorf 2-basis is the Fibonacci sequence $(b_j^{(2)} = F_{j+1})$, and one may view the work in this section as an attempt to generalize the results in Section 2. Table 2 shows part of the 3-order Zeckendorf array.

Next, we generalize the shift function $f: N \to N$ as defined in Section 2. In terms of *m*-order Zeckendorf representations, the generalized function $f^{(m)}$ is given as follows:

if
$$n = \sum_{h=1}^{\infty} c_h b_h^{(m)}$$
, then $f^{(m)}(n) = \sum_{h=1}^{\infty} c_h b_{h+1}^{(m)}$.

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TABLE 2. The 3rd-Order Zeckendorf Array

1	2	3	4	6	9	13	19	28	41	60	
5	8	12	17	25	37	54	79	116	170	249	
7	11	16	23	34	50	73	107	157	230	337	
10	15	22	32	47	69	101	148	217	318	466	
14	21	31	45	66	97	142	208	306	448	656	
18	27	40	58	85	125	183	268	393	576	844	
20	30	44	64	94	138	202	296	434	636	932	
24	36	53	77	113	166	243	356	522	765	1121	
26	39	57	83	122	179	262	384	563	825	1209	
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Lemma 1: The shift function $f^{(m)}$ is a strictly increasing function.

Proof: As a first inductive step, we have $2 = f^{(m)}(1) < 3 = f^{(m)}(2)$. Assume $K \ge 2$ and that for every $k_1 < K$ it is true that $f^{(m)}(k_1) < f^{(m)}(K)$. Let k be any positive integer satisfying $k \le K$. Let

$$h_0 = \max_{b_j \le K+1} \{j\} \text{ and } h_1 = \max_{b_j \le k} \{j\}.$$

<u>**Case 1</u>:** $h_1 < h_0$. Let $x = b_{h_0-1}^{(m)} + b_{h_0-1-m}^{(m)} + b_{h_0-1-2m}^{(m)} + \dots + b_s^{(m)}$, where $s = h_0 - 1 - [\frac{h_0-2}{m}]m$. Since $x + 1 = b_{h_0}^{(m)}$, we have $k \le x < K + 1$ and</u>

$$f^{(m)}(k) \leq f^{(m)}(x) = b_{h_0}^{(m)} + b_{h_0-m}^{(m)} + b_{h_0-2m}^{(m)} + \dots + b_{s+1}^{(m)} < b_{h_0+1}^{(m)} \leq b_{h_1}^{(m)} \leq f^{(m)}(K+1).$$

<u>Case 2</u>: $h_1 = h_0$. Here, $k - b_{h_0}^{(m)} < K + 1 - b_{h_0}^{(m)}$. By the induction hypothesis,

$$f^{(m)}(k-b_{h_0}^{(m)}) < f^{(m)}(K+1-b_{h_0}^{(m)}).$$

Then

$$f^{(m)}(k) = f^{(m)}(k - b_{h_0}^{(m)}) + b_{h_0+1}^{(m)} < f^{(m)}(K + 1 - b_{h_0}^{(m)}) + b_{h_0+1}^{(m)} = f^{(m)}(K + 1).$$

In both cases, $f^{(m)}(k) < f^{(m)}(K+1)$ for all k < K+1, so that we conclude that $f^{(m)}$ is strictly increasing. \Box

Theorem 3: The first column of the *m*-order Zeckendorf array determines all of the array by the recurrences (2) for all $i \ge 1$ and $j \ge 1$.

Proof: The proof is analogous to that of Theorem 1 and is omitted.

Theorem 4: For every $m \ge 2$, the *m*-order Zeckendorf array is an interspersion.

Proof: Of the four properties that define an interspersion (as introduced in [7]), it is clear that $Z^{(m)}$ satisfies the first three: every positive integer occurs exactly once in $Z^{(m)}$; every row of $Z^{(m)}$ is increasing; and every column of $Z^{(m)}$ is increasing. To prove the fourth property, suppose *i*, *j*, *i'*, *j'* are indices for which

$$z(i, j) < z(i', j') < z(i, j+1).$$

Then, by Lemma 1,

$$f^{(m)}(z(i, j)) < f^{(m)}(z(i', j')) < f^{(m)}(z(i, j+1)),$$

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so that, by Theorem 3,

$$z(i, j+1) < z(i', j'+1) < z(i, j+2),$$

as required. \Box

Consider the recurrence (1) which defines the Wythoff array W in terms of the golden mean, α . Since α is the real root of the characteristic polynomial $x^2 - x - 1$ of the recurrence relation for the row sequences of W, one must wonder if the real root $\alpha^{(m)}$ of $x^m - x^{m-1} - 1$ can, in some manner comparable to (1), be used to generate rows of the *m*-order Zeckendorf array. The answer seems to be no, although certain "higher-order" Wythoff-like arrays have been investigated (see [3], [6]).

However, Beatty's theorem leads to conjectures about column 1 of $Z^{(m)}$. It appears that each row of $Z^{(m)}$ has "slope" $\alpha^{(m)}$, so that the complement of column 1, ordered as an increasing sequence, is comparable to the set of numbers $[i\alpha^{(m)}]$. Beatty's theorem then suggests that column 1 is "close to" the sequence $\{c_i\}$ given by $c_i = \lfloor \frac{i\alpha^{(m)}}{\alpha^{(m)}-1} \rfloor$. For example, taking m = 3, let $s_i = \lfloor \frac{\alpha^{(3)}i}{\alpha^{(3)}-1} \rfloor - \lfloor \alpha^{(3)} \rfloor$. Let x_i denote the *i*th number in column 1 of $Z^{(3)}$. We conjecture that $|z_i - s_i| \le 1$ for all $i \ge 1$.

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