# THE ZECKENDORF ARRAY EQUALS THE WYTHOFF ARRAY 

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## 1. INTRODUCTION

It is well known that every $n$ in the set $N$ of positive integers is uniquely a sum of nonconsecutive Fibonacci numbers. This sum $n$ is known as the Zeckendorf representation of $n$. We arrange these representations to form the Zeckendorf array $Z=Z(i, j)$ as follows: column $j$ of $Z$ is the increasing sequence of all $n$ in whose Zeckendorf representation the least term is $F_{j+1}$. The first row of $Z$ therefore consists of Fibonacci numbers:

$$
z(1,1)=1=F_{2} \quad z(1,2)=2=F_{3} \quad z(1,3)=3=F_{4} \quad \cdots \quad z(1, j)=F_{j+1} \cdots,
$$

and the second row begins with the numbers $4=3+1,7=5+2,11=8+3$, and $18=13+5$. The reader is urged to write down several terms of the next two rows before reading further.

The Wythoff array, $W=W(i, j)$, partly shown in Table 1, was introduced by David R. Morrison [9], in connection with Wythoff pairs, which are the winning pairs of numbers in Wythoff's game. (See, for example, [2], [12]). Morrison proved several interesting things about $W$ : every positive integer $n$ occurs exactly once in $W$, as does every Wythoff pair; every row is a (generalized) Fibonacci sequence [i.e., $w(i, j)=w(i, j-1)+w(i, j-2)$ for every $i \geq 1$ and $j \geq 3$ ]. In fact, Morrison proved that, in a sense, every positive Fibonacci sequence of integers is a row of $W$.

## TABLE 1. The Wythoff Array

| 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | $\ldots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 | $\ldots$ |
| 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 | 754 |  |
| 9 | 15 | 24 | 39 | 63 | 102 | 165 | 267 | 432 | 699 | 1131 |  |
| 12 | 20 | 32 | 52 | 84 | 136 | 220 | 356 | 576 | 932 | 1508 |  |
| 14 | 23 | 37 | 60 | 97 | 157 | 254 | 411 | 665 | 1076 | 1741 |  |
| 17 | 28 | 45 | 73 | 118 | 191 | 309 | 500 | 809 | 1309 | 2118 |  |
| 19 | 31 | 50 | 81 | 131 | 212 | 343 | 555 | 898 | 1453 | 2351 |  |
| 22 | 36 | 58 | 94 | 152 | 246 | 398 | 644 | 1042 | 1686 | 2728 |  |
| $\vdots$ |  |  |  |  |  |  |  |  |  |  |  |

Morrison also proved that the first column of $W$ is given by $w(i, 1)=[i \alpha]+i-1$, where $\alpha=(1+\sqrt{5}) / 2$. The rest of $W$ is then given inductively by

$$
w(i, j+1)=\left\{\begin{array}{ll}
{[\alpha w(i, j)]+1} & \text { if } j \text { is odd, }  \tag{1}\\
{[\alpha w(i, j)]} & \text { if } j \text { is even, }
\end{array} \text { for } i=1,2,3, \ldots .\right.
$$

## 2. SHIFTING SUBSCRIPTS: $\boldsymbol{F}_{\boldsymbol{n}+1} \rightarrow \boldsymbol{F}_{\boldsymbol{n}+2}$

We define a shift function $f: N \rightarrow N$ in terms of Zeckendorf representations:

$$
\text { if } n=\sum_{h=1}^{\infty} c_{h} F_{h+1} \text {, then } f(n)=\sum_{h=1}^{\infty} c_{h} F_{h+2} \text {. }
$$

Lemma 1: The shift function $f$ is a strictly increasing function.
We shall prove Lemma 1 in a more general form in Section 3.
Theorem 1: The first column of the Zeckendorf array $Z$ determines all of $Z$ by the recurrences

$$
\begin{equation*}
z(i, j+1)=f(z(i, j)) \tag{2}
\end{equation*}
$$

for all $i \geq 1$ and $j \geq 1$.
Proof: We have $z(1, j)=F_{j+1}$ for all $j \geq 1$, so that row 1 of $Z$ is determined by $z(1,1)=1$ and $f$. Assume $k \geq 1$ and that (2) holds for all $j \geq 1$, for all $i \leq k$. Write the Zeckendorf representation of $z(k+1,1)$ as $z(k+1,1)=\sum_{h=1}^{\infty} c_{h} F_{h+1}$, noting that the following conditions hold:
(i) $c_{1}=1$;
(ii) $c_{h} \in\{0,1\}$ for every $h$ in $N$;
(iii) for every $h$ in $N$, if $c_{h}=1$ then $c_{h+1}=0$;
(iv) there exists $n$ in $N$ such that $c_{h}=0$ for every $h \geq n$.

Let $m=f(z(k+1,1))$. Then the representation $\sum_{h=1}^{\infty} c_{h}^{\prime} F_{h+1}$, where $c_{1}^{\prime}=0$ and $c_{h}^{\prime}=c_{h-1}$ for all $h \geq 2$, is the Zeckendorf representation of $m$. Also, $m$ is in column 2 of $Z$, since $c_{1}^{\prime}=0$ and $c_{2}^{\prime}=1$. By the induction hypothesis, $z(i, 2)=f(z(i, 1))$ for $i=1,2, \ldots, k$, and since column 2 is an increasing sequence, $m$ must lie in a row numbered $\geq k+1$ by Lemma 1. We shall show that this row number cannot be $>k+1$.

Suppose $m>z(k+1,2)$ and let the Zeckendorf representation for $z(k+1,2)$ be $\sum_{h=1}^{\infty} d_{h}^{\prime} F_{h+1}$. Then the number $q=\sum_{h=1}^{\infty} d_{h} F_{h+1}$, where $d_{h}=d_{h+1}^{\prime}$ for $h \geq 1$, is the Zeckendorf representation for a number having $d_{1}=1$, so that this number lies in column 1 of $Z$. It is not one of the first $k$ terms, and it is not $z(k+1,1)$ since its sequel in row $k+1$ is not $m$. Therefore, $q=z(K, 1)$ for some $K \geq k+2$. We now have $z(k+1,1)<q$ and $f(q)<f(z(k+1,1))$, a contradiction to Lemma 1 . Therefore, $z(k+1,2)=f(z(k+1,1))$.

Let $j \geq 2$ and suppose that $z(k+1, j)=f(z(k+1, j-1))$. The argument just used for $j=2$ applies here in the same way, giving $z(k+1, j+1)=f(z(k+1, j))$. The induction on $j$ is finished, so that (2) holds for all $j \geq 1$ for $i=k+1$. Consequently, the induction on $k$ is finished, so that (2) holds throughout $Z$.

## Lemma 2:

$$
f(n)= \begin{cases}{[\alpha n]+1} & \text { if } n \text { is in an odd numbered column of } Z \\ {[\alpha n]} & \text { if } n \text { is in an even numbered column of } Z .\end{cases}
$$

Proof: The fact that the continued fraction for $\alpha$ is $[1,1,1, \ldots]$ leads as in $[10, \mathrm{p} .10]$ to the well-known inequality

$$
\frac{1}{F_{h+2}}<\left|\alpha F_{h}-F_{h+1}\right|<\frac{1}{F_{h+1}}
$$

for $h=1,2,3, \ldots$, and this in turn implies

$$
\begin{equation*}
\frac{1}{F_{h+2}}<\left\{\alpha F_{h}\right\}<\frac{1}{F_{h+1}} \text { for odd } h \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
-\frac{1}{F_{h+1}}<\left\{\alpha F_{h}\right\}-1<-\frac{1}{F_{h+2}} \text { for even } h . \tag{4}
\end{equation*}
$$

Write the Zeckendorf representation of $n$ as indicated by the sum

$$
\begin{equation*}
n=c_{1} F_{2}+c_{2} F_{3}+c_{3} F_{4}+\cdots \tag{5}
\end{equation*}
$$

Then

$$
\begin{equation*}
f(n)=c_{1} F_{3}+c_{2} F_{4}+c_{3} F_{5}+\cdots . \tag{6}
\end{equation*}
$$

Also,

$$
\begin{aligned}
n \alpha & =c_{1} \alpha F_{2}+c_{2} \alpha F_{3}+c_{3} \alpha F_{4}+\cdots \\
& =c_{1}\left(F_{3}+\left\{\alpha F_{2}\right\}-1\right)+c_{2}\left(F_{4}+\left\{\alpha F_{3}\right\}\right)+c_{3}\left(F_{5}+\left\{\alpha F_{4}\right\}-1\right)+\cdots \\
& =f(n)+\mathscr{S}_{1}(n)+\mathscr{S}_{2}(n),
\end{aligned}
$$

where

$$
\mathscr{S}_{1}(n)=c_{1}\left(\left\{\alpha F_{2}\right\}-1\right)+c_{3}\left(\left\{\alpha F_{4}\right\}-1\right)+c_{5}\left(\left\{\alpha F_{6}\right\}-1\right)+\cdots
$$

and

$$
\mathscr{S}_{2}(n)=c_{2}\left\{\alpha F_{3}\right\}+c_{4}\left\{\alpha F_{5}\right\}+c_{6}\left\{\alpha F_{7}\right\}+\cdots .
$$

Case 1: $n$ is an even numbered column of $Z$. In this case, the least nonzero coefficient $c_{H}$ in (5) has an even index $H$, so that

$$
\begin{aligned}
\mathscr{S}_{1}(n)+\mathscr{C}_{2}(n) & =\left\{\alpha F_{H+1}\right\}+\text { other terms } \\
& \leq\left\{\alpha F_{H+1}\right\}+\left\{\alpha F_{H+3}\right\}+\cdots<\frac{1}{F_{H+2}}+\frac{1}{F_{H+4}}+\frac{1}{F_{H+6}}+\cdots \text { by }(3) \\
& \leq \frac{1}{2}+\frac{1}{5}+\frac{1}{13}+\cdots<\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1 .
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathscr{S}_{1}(n)+\mathscr{S}_{2}(n) & \geq\left\{\alpha F_{H+1}\right\}+\left(-1+\left\{\alpha F_{H+4}\right\}\right)+\left(-1+\left\{\alpha F_{H+6}\right\}\right)+\cdots \\
& >\frac{1}{F_{H+3}}-\frac{1}{F_{H+5}}-\frac{1}{F_{H+7}}-\cdots \text { by }(3) \text { and }(4) \\
& \geq \frac{1}{F_{H+3}}-\frac{1}{F_{H+5}}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) \geq \frac{1}{F_{H+3}}-\frac{2}{F_{H+5}}>0 .
\end{aligned}
$$

The conclusion in Case 1 is that $f(n)=[n \alpha]$.
Case 2: $n$ is an odd numbered column of $Z$. Then the least nonzero coefficient $c_{H}$ in (5) has an odd index $H$, and

$$
\begin{aligned}
\mathscr{S}_{1}(n)+\mathscr{S}_{2}(n) & =-1+\left\{\alpha F_{H+1}\right\}+\text { other terms } \\
& \leq-1+\left\{\alpha F_{H+1}\right\}+\left\{\alpha F_{H+4}\right\}+\left\{\alpha F_{H+6}\right\}+\cdots \\
& <-\frac{1}{F_{H+3}}+\frac{1}{F_{H+5}}+\frac{1}{F_{H+7}}+\cdots \text { by (3) and (4) } \\
& \leq-\frac{1}{F_{H+3}}+\frac{2}{F_{H+5}}<0 .
\end{aligned}
$$

Also

$$
\begin{aligned}
\mathscr{S}_{1}(n)+\mathscr{S}_{2}(n) & \geq-1+\left\{\alpha F_{H+1}\right\}+\left(-1+\left\{\alpha F_{H+3}\right\}\right)+\left(-1+\left\{\alpha F_{H+5}\right\}\right)+\cdots \\
& >-\frac{1}{F_{H+2}}-\frac{1}{F_{H+4}}-\frac{1}{F_{H+6}}-\cdots \text { by }(4) \\
& >-\frac{2}{F_{H+2}} \geq-1 .
\end{aligned}
$$

The conclusion in Case 2 is that $f(n)=[n \alpha]+1$.
Theorem 2: The Zeckendorf array equals the Wythoff array.
Proof: Let $C$ be the set of numbers in the first column of $Z$. Let $S$ be the complement of $C$ in the set of positive integers. Let $\left\{s_{n}\right\}$ be the sequence obtained by arranging the elements of $S$ in increasing order. It is known [4] that this sequence of one-free Zeckendorf sums is given by $s_{n}=[(n+1) \alpha]-1, n=1,2,3, \ldots$. We shall apply Beatty's theorem (see [1], [11]) on complementary sequences of positive integers to prove that $z(i, 1)=[i \alpha]+i-1$ : first, $\frac{1}{\alpha}+\frac{1}{\alpha+1}=1$, so that, by Beatty's theorem, the sequences $[n \alpha]$ and $[i \alpha]+i$ are complementary; this implies that the sets $\{[(n+1) \alpha]\}$ and $\{[i \alpha]+i\} \cup\{1\}$ partition $N$, which in turn implies that the sequences $s_{n}$ and $z(i, 1)$ are complementary. Since $w(i, 1)=[i \alpha]+i-1$, we have $z(i, 1)=w(i, 1)$. Now the recurrence (1) together with Theorem 1 and Lemma 2 imply that $Z=W$.

## 3. HIGHER-ORDER ZECKENDORF ARRAYS

Let $m$ be an integer $\geq 2$. Define a sequence $\left\{s_{i}\right\}$ inductively, as follows:

$$
\begin{array}{ll}
s_{i}=1 & \text { for } i=1,2,3, \ldots, m \\
s_{i}=s_{i-1}+s_{i-m} & \text { for } i=m+1, m+2, \ldots
\end{array}
$$

and define the Zeckendorf m-basis as the sequence $\left\{b_{j}^{(m)}\right\}$, where $b_{j}^{(m)}=s_{m+j-1}$ for all $j$ in $N$. It is proved in [5] and probably elsewhere that every $n$ in $N$ is uniquely a sum

$$
\begin{equation*}
b_{i_{1}}^{(m)}+b_{i_{2}}^{(m)}+\cdots+b_{i_{v}}^{(m)}, \text { where } i_{t}-i_{u} \geq m \text { whenever } t>u \tag{7}
\end{equation*}
$$

We call the sum in (7) the $m$-order Zeckendorf representation of $n$, and we define the $m$-order Zeckendorf array $Z^{(m)}=Z^{(m)}(i, j)$ as follows: column $j$ of $Z^{(m)}$ is the increasing sequence of all $n$ in whose $m$-order Zeckendorf representation the least term is $b_{j}^{(m)}$. The first row of $Z^{(m)}$ is the Zeckendorf $m$-basis. Of course, the Zeckendorf 2-basis is the Fibonacci sequence $\left(b_{j}^{(2)}=F_{j+1}\right)$, and one may view the work in this section as an attempt to generalize the results in Section 2. Table 2 shows part of the 3-order Zeckendorf array.

Next, we generalize the shift function $f: N \rightarrow N$ as defined in Section 2. In terms of $m$ order Zeckendorf representations, the generalized function $f^{(m)}$ is given as follows:

$$
\text { if } n=\sum_{h=1}^{\infty} c_{h} b_{h}^{(m)}, \text { then } f^{(m)}(n)=\sum_{h=1}^{\infty} c_{h} b_{h+1}^{(m)}
$$

TABLE 2. The 3rd-Order Zeckendorf Array

| 1 | 2 | 3 | 4 | 6 | 9 | 13 | 19 | 28 | 41 | 60 | $\ldots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 5 | 8 | 12 | 17 | 25 | 37 | 54 | 79 | 116 | 170 | 249 | $\ldots$ |
| 7 | 11 | 16 | 23 | 34 | 50 | 73 | 107 | 157 | 230 | 337 |  |
| 10 | 15 | 22 | 32 | 47 | 69 | 101 | 148 | 217 | 318 | 466 |  |
| 14 | 21 | 31 | 45 | 66 | 97 | 142 | 208 | 306 | 448 | 656 |  |
| 18 | 27 | 40 | 58 | 85 | 125 | 183 | 268 | 393 | 576 | 844 |  |
| 20 | 30 | 44 | 64 | 94 | 138 | 202 | 296 | 434 | 636 | 932 |  |
| 24 | 36 | 53 | 77 | 113 | 166 | 243 | 356 | 522 | 765 | 1121 |  |
| 26 | 39 | 57 | 83 | 122 | 179 | 262 | 384 | 563 | 825 | 1209 |  |
| $\vdots$ |  |  |  |  |  |  |  |  |  |  |  |

Lemma 1: The shift function $f^{(m)}$ is a strictly increasing function.
Proof: As a first inductive step, we have $2=f^{(m)}(1)<3=f^{(m)}(2)$. Assume $K \geq 2$ and that for every $k_{1}<K$ it is true that $f^{(m)}\left(k_{1}\right)<f^{(m)}(K)$. Let $k$ be any positive integer satisfying $k \leq K$. Let

$$
h_{0}=\max _{b_{j} \leq K+1}\{j\} \text { and } h_{1}=\max _{b_{j} \leq k}\{j\}
$$

Case 1: $\quad h_{1}<h_{0}$. Let $x=b_{h_{0}-1}^{(m)}+b_{h_{0}-1-m}^{(m)}+b_{h_{0}-1-2 m}^{(m)}+\cdots+b_{s}^{(m)}$, where $s=h_{0}-1-\left[\frac{h_{0}-2}{m}\right] m$. Since $x+1=b_{h_{0}}^{(m)}$, we have $k \leq x<K+1$ and

$$
f^{(m)}(k) \leq f^{(m)}(x)=b_{h_{0}}^{(m)}+b_{h_{0}-m}^{(m)}+b_{h_{0}-2 m}^{(m)}+\cdots+b_{s+1}^{(m)}<b_{h_{0}+1}^{(m)} \leq b_{h_{1}}^{(m)} \leq f^{(m)}(K+1) .
$$

Case 2: $h_{1}=h_{0}$. Here, $k-b_{h_{0}}^{(m)}<K+1-b_{h_{0}}^{(m)}$. By the induction hypothesis,

$$
f^{(m)}\left(k-b_{h_{0}}^{(m)}\right)<f^{(m)}\left(K+1-b_{h_{0}}^{(m)}\right) .
$$

Then

$$
f^{(m)}(k)=f^{(m)}\left(k-b_{h_{0}}^{(m)}\right)+b_{h_{0}+1}^{(m)}<f^{(m)}\left(K+1-b_{h_{0}}^{(m)}\right)+b_{h_{0}+1}^{(m)}=f^{(m)}(K+1)
$$

In both cases, $f^{(m)}(k)<f^{(m)}(K+1)$ for all $k<K+1$, so that we conclude that $f^{(m)}$ is strictly increasing.

Theorem 3: The first column of the $m$-order Zeckendorf array determines all of the array by the recurrences (2) for all $i \geq 1$ and $j \geq 1$.

Proof: The proof is analogous to that of Theorem 1 and is omitted.
Theorem 4: For every $m \geq 2$, the $m$-order Zeckendorf array is an interspersion.
Proof: Of the four properties that define an interspersion (as introduced in [7]), it is clear that $Z^{(m)}$ satisfies the first three: every positive integer occurs exactly once in $Z^{(m)}$; every row of $Z^{(m)}$ is increasing; and every column of $Z^{(m)}$ is increasing. To prove the fourth property, suppose $i, j, i^{\prime}, j^{\prime}$ are indices for which

$$
z(i, j)<z\left(i^{\prime}, j^{\prime}\right)<z(i, j+1)
$$

Then, by Lemma 1,

$$
f^{(m)}(z(i, j))<f^{(m)}\left(z\left(i^{\prime}, j^{\prime}\right)\right)<f^{(m)}(z(i, j+1))
$$

so that, by Theorem 3,

$$
z(i, j+1)<z\left(i^{\prime}, j^{\prime}+1\right)<z(i, j+2),
$$

as required.
Consider the recurrence (1) which defines the Wythoff array $W$ in terms of the golden mean, $\alpha$. Since $\alpha$ is the real root of the characteristic polynomial $x^{2}-x-1$ of the recurrence relation for the row sequences of $W$, one must wonder if the real root $\alpha^{(m)}$ of $x^{m}-x^{m-1}-1$ can, in some manner comparable to (1), be used to generate rows of the $m$-order Zeckendorf array. The answer seems to be no, although certain "higher-order" Wythoff-like arrays have been investigated (see [3], [6]).

However, Beatty's theorem leads to conjectures about column 1 of $Z^{(m)}$. It appears that each row of $Z^{(m)}$ has "slope" $\alpha^{(m)}$, so that the complement of column 1, ordered as an increasing sequence, is comparable to the set of numbers $\left[i \alpha^{(m)}\right]$. Beatty's theorem then suggests that column 1 is "close to" the sequence $\left\{c_{i}\right\}$ given by $c_{i}=\left\{\frac{i \alpha^{(m)}}{\alpha^{(m)}-1}\right\rfloor$. For example, taking $m=3$, let $s_{i}=$ $\left\lfloor\frac{\alpha^{(3)} j}{\alpha^{(3)}-1}\right\rfloor-\left[\alpha^{(3)}\right]$. Let $x_{i}$ denote the $i^{\text {th }}$ number in column 1 of $Z^{(3)}$. We conjecture that $\left|z_{i}-s_{i}\right| \leq 1$ for all $i \geq 1$.

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