PIERCE EXPANSIONS OF RATIOS OF FIBONACCI AND LUCAS NUMBERS AND POLYNOMIALS

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INTRODUCTION

The connection between the Euclidean algorithm for determining the greatest common divisor of two positive integers a and b and the continued fraction expansion of the rational number a/b is well known. As Lamé [9] observed, two successive Fibonacci numbers F_{n-1} and F_n provide a pair of integers for which the Euclidean algorithm takes as long as possible to terminate, in the sense that (F_{n-1}, F_n) takes as long or longer than any pair (a, b) with $b > a > 1$ and $b \le F_n$. Analogous results hold if arithmetic is done in $\mathbb{Q}[x]$ or $\mathbb{F}_q[x]$, for $\mathbb{F}_q[x]$ the finite field with $q = p^a$ elements [4], [6], [7], [10].

One can view the continued fraction expansion more generally as an association with a real number of a sequence of positive integers, the sequence being finite if and only if the real number is rational. Other methods exist to accomplish the same task. Two in particular of interest are the so-called Engel expansion and the Pierce expansion. Each arises from an iterated division algorithm, but the roles of successive dividends and divisors are played by different elements than in the Euclidean algorithm.

In particular, for $1 \le a \le b$ integers, the Pierce expansion of a/b is the unique representation

$$\frac{a}{b} = \frac{1}{x_1} - \frac{1}{x_1x_2} + \frac{1}{x_1x_2x_3} - \cdots + \frac{(-1)^{n-1}}{x_1x_2 \ldots x_n}, \tag{1}$$

where the x_i are integers with $1 \le x_1 < x_2 < \cdots < x_n$. Successive x_i may be obtained via the division algorithm. Write $b = qa + r$ with q and r nonnegative integers and $r < a$. Then $a = \frac{b-r}{q}$ and so $\frac{a}{b} = \frac{1}{q} - \frac{1}{q}\left(\frac{r}{b}\right)$. Thus, $x_1 = q$, and the procedure may be applied again to the fraction r/b. The iteration stops when $r = 0$, which must happen after at most a steps (see [11]). A convenient notation for this expansion is

$$\frac{a}{b} = \{x_1, x_2, x_3, \ldots, x_n\}.$$

The Engel expansion is a similar expansion with all positive terms. Thus, for $1 \le a \le b$ integers, it is the unique representation of the form

$$\frac{a}{b} = \frac{1}{y_1} + \frac{1}{y_1y_2} + \frac{1}{y_1y_2y_3} + \cdots + \frac{1}{y_1y_2 \cdots y_n}, \tag{2}$$

where the y_i are integers with $1 \le y_1 \le y_2 \le \cdots \le y_n$. Here one would iterate the version of the division algorithm with negative remainders. Thus, $b = qa + r = (q+1)a - (a-r)$ and, hence, $a = \frac{b+(a-r)}{q+1}$ gives $\frac{a}{b} = \frac{1}{q+1} + \frac{1}{q+1}\left(\frac{a-r}{b}\right)$. The procedure is applicable again until $r = 0$. This expansion is frequently denoted

$$\frac{a}{b} = (y_1, y_2, y_3, \ldots, y_n).$$

Maximal lengths and other properties of Pierce and Engel expansions have been studied in [2], [11], [13], and [14].

In the case of polynomial rings, the appropriate measure of the size of the remainder is given by its degree, so that signs are no longer relevant and there is no distinguishing the Pierce and Engel expansions. For the Fibonacci polynomials [1] defined by

$$F_1(x) = 1,\ F_2(x) = x,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \ \text{ for } n \geq 2,$$

and the Lucas polynomials given by

$$L_0(x) = 2,\ L_1(x) = x,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x) \ \text{ for } n \geq 1,$$

there are some especially attractive continued fraction expansions. In particular,

$$\frac{F_{n-1}(x)}{F_n(x)} = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{\ddots + \cfrac{1}{x}}}}, \tag{3}$$

where there are $n-1$ occurrences of x in (3), and

$$\frac{L_{n-1}(x)}{L_n(x)} = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{\ddots + \cfrac{1}{x/2}}}}, \tag{4}$$

where the continued fraction in (4) has n x's in its expansion.

Motivated by these expansions, we consider the Pierce-Engel expansions of these rational functions. In contrast to the longest possible expansions in their continued fraction expansions, the Pierce-Engel expansions are predictably short. For some special values of n they are especially short and elegant.

There are also regularities to note in the Pierce expansions of the rational numbers F_{n-1}/F_n and L_{n-1}/L_n. One such follows from a general result of Shallit [13], and we establish others in the last section.

EXPANSIONS OF FIBONACCI AND LUCAS POLYNOMIAL QUOTIENTS

We are most interested in the quotients $F_{n-1}(x)/F_n(x)$ and $L_{n-1}(x)/L_n(x)$, although the theorems we use apply more generally. Since in the limit we have

$$\lim_{n\to\infty} \frac{F_{n-1}(1/x)}{F_n(1/x)} = \frac{-1+\sqrt{1+4x^2}}{2x},$$

as n increases there are ever more terms incorporated in the infinite Pierce/Engel expansion of this function, shown in [3] to be

$$\frac{1}{L_1(z)}-\frac{1}{L_1(z)L_2(z)}-\frac{1}{L_1(z)L_2(z)L_4(z)}-\frac{1}{L_1(z)L_2(z)L_4(z)L_8(z)}-\cdots,$$

where $z=x^{-1}$. This particular expansion is also a concrete example of the Engel-type expansions for power series developed in [8]. This limiting case sets the pattern for the finite expansions of rational functions in the variable x. Using the notation $(a, b, c, d, \ldots)$ introduced earlier for the expansion

$$\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\cdots,$$

the finite expansion beginning

$$\frac{1}{L_1(x)}-\frac{1}{L_1(x)L_2(z)}-\frac{1}{L_1(x)L_2(x)L_4(x)}-\frac{1}{L_1(x)L_2(x)L_4(x)L_8(x)}-\cdots$$

can be written more compactly as $(L_1, -L_2, L_4, L_8, \ldots)$. Later we also allow more complicated expressions involving Lucas polynomials as entries.

It is possible to write an alternate representation in terms of the Chebyshev polynomials $C_n(x)=2T_n(x/2)$, where

$$T_0(x)=1,\ T_1(x)=x,\ T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)\ \text{ for } n\geq 1,$$

since $C_n(x)=(-i)^n L_n(ix)$.

The form of the expansions follows from two general results in [1], which we state as lemmas.

Lemma 1: Whenever a Fibonacci polynomial $F_m(x)$ is divided by a Fibonacci polynomial $F_{m-k}(x)$, $m\neq k$, of lesser or equal degree, the remainder is always a Fibonacci polynomial or the negative of a Fibonacci polynomial, and the quotient is a sum of Lucas polynomials whenever the division is not exact. Explicitly, for $p\geq 1$:

(i) the remainder is $\pm F_{(2p-1)m-2kp}(x)$ when

$$\frac{2p|m|}{2p+1}>|k|>\frac{(2p-2)|m|}{2p-1};$$

(ii) the quotient is $\pm L_k(x)$ when $|k|<2|m|/3$;

(iii) the quotient is given by

$$Q_p(x)=\sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{(2i+1)k-2im}(x)$$

for m, k, and p as in (i), and by $Q_p(x)+(-1)^{p(m-k)}$ if $k=2pm/(2p+1)$;

(iv) the division is exact when $k=2pm/(2p+1)$ or $k=(2p-1)m/2p$.

Lemma 2: Whenever a Lucas polynomial $L_m(x)$ is divided by a Lucas polynomial $L_{m-k}(x)$, $m \neq k$, of lesser degree, a nonzero remainder is always a Lucas polynomial or the negative of a Lucas polynomial. Explicitly:

(i) nonzero remainders have the form $\pm L_{(2p-1)m-2pk}(x)$ when

$$\frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1};$$

(ii) if $|k| < 2|m|/3$, the quotient is $\pm L_k(x)$;

(iii) the division is exact when $k = 2pm/(2p+1)$, $p \neq 0$.

These lemmas apply to give

Theorem 3: Any quotient of Fibonacci polynomials or Lucas polynomials has a (finite) Pierce-Engel expansion in which every entry is expressible as a linear combination of Lucas polynomials with coefficients 0 or ± 1. In the case $F_{n-1}(x)/F_n(x)$ or $L_{n-1}(x)/L_n(x)$, there are at least $m = \lfloor \log_2 n \rfloor$ entries, and the first $\lfloor \log_2 n \rfloor$ entries are $(L_1, -L_2, L_4, \ldots, L_{2^{m-1}})$.

Proof: The Pierce-Engel expansion for quotients of Fibonacci polynomials comes from the sequence of identities

$$\begin{aligned} F_n(x) &= L_k(x)F_{n-k}(x) + (-1)^{k+1}F_{n-2k}(x) \\ F_n(x) &= L_{2k}(x)F_{n-2k}(x) - F_{n-4k}(x) \\ F_n(x) &= L_{4k}(x)F_{n-4k}(x) - F_{n-8k}(x) \\ &\vdots, \end{aligned}$$

which may be continued as long as the last subscript remains nonnegative. These identities may be read as special cases of Lemma 1. Lemma 2 provides similar identities for Lucas polynomials. A negative subscript is replaced by a positive subscript via the identity $F_m(x) = (-1)^{m+1}F_{-m}(x)$. Then

$$\begin{aligned} \frac{F_{n-k}(x)}{F_n(x)} &= \frac{1}{L_k(x)}\left(1 + (-1)^k \frac{F_{n-2k}(x)}{F_n(x)}\right) \\ &= \frac{1}{L_k(x)}\left(1 + \frac{(-1)^k}{L_{2k}(x)}\left(1 - \frac{F_{n-4k}(x)}{F_n(x)}\right)\right) \\ &= \cdots \\ &= \left(L_k(x), (-1)^k L_{2k}(x), L_{4k}(x), \ldots\right). \end{aligned}$$

Table 1 on the following page gives Pierce-Engel expansions of some rational functions for small values of n.

The next theorem was obtained in [3] and [16]. The technique of proof can be modified to provide several other similar relations, which are collected in the theorem thereafter.

Theorem 4: For $n \geq 1$, $\dfrac{F_{2^n-1}(x)}{F_{2^n}(x)} = (L_1, -L_2, L_4, \ldots, L_{2^{n-1}})$

TABLE 1. Expansions of Quotients of Fibonacci Polynomials

n	Pierce-Engel Expansion of $F_{n-1}(x)/F_n(x)$
2	L_1
3	$L_1, -(L_2-1)$
4	$L_1, -L_2$
5	$L_1, -L_2, L_4-L_2+1$
6	$L_1, -L_2, L_4+1$
7	$L_1, -L_2, L_4, L_6-L_4+L_2-1$
8	$L_1, -L_2, L_4$
9	$L_1, -L_2, L_4, L_8-L_6+L_4-L_2+1$
10	$L_1, -L_2, L_4, L_8+L_4+1$
11	$L_1, -L_2, L_4, L_8-L_2, -(L_{10}-L_8+L_6-L_4+L_2-1)$
12	$L_1, -L_2, L_4, L_8+1$
13	$L_1, -L_2, L_4, L_8, L_{10}-L_4, -(L_{12}-L_{10}+L_8-L_6+L_4-L_2+1)$
14	$L_1, -L_2, L_4, L_8, -(L_{12}+L_8+L_4+1)$
15	$L_1, -L_2, L_4, L_8, L_{14}-L_{12}+L_{10}-L_8+L_6-L_4+L_2-1$
16	$L_1, -L_2, L_4, L_8$
17	$L_1, -L_2, L_4, L_8, L_{16}-L_{14}+L_{12}-L_{10}+L_8-L_6+L_4-L_2+1$
18	$L_1, -L_2, L_4, L_8, L_{16}+L_{12}+L_8+L_4+1$

Theorem 5: For $n \geq 1$, $\dfrac{F_{3\cdot 2^n-1}(x)}{F_{3\cdot 2^n}(x)} = (L_1, -L_2, L_4, \ldots, L_{2^n}, L_{2^{n+1}}+1)$.

For $n \geq 2$, $\dfrac{F_{2^n}(x)}{F_{2^n+1}(x)} = \left(L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, \sum_{i=0}^{2^{n-1}}(-1)^i L_{2i} - 1\right)$.

For $n \geq 3$, $\dfrac{F_{2^n-2}(x)}{F_{2^n-1}(x)} = \left(L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, -\sum_{i=0}^{2^{n-1}-1}(-1)^i L_{2i} + 1\right)$.

There are, in addition, dual results for Lucas polynomials. A brief table of Lucas polynomial expansions follows (see Table 2), and a general theorem (Theorem 6) makes explicit some of the patterns apparent in the table. Other patterns may be noted in the tables as well.

Theorem 6: For $n \geq 2$, $\dfrac{L_{2^n-1}(x)}{L_{2^n}(x)} = (L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, L_{2^n}/2)$.

For $n \geq 2$, $\dfrac{L_{2^n}(x)}{L_{2^n+1}(x)} = \left(L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, \sum_{i=0}^{2^{n-1}} L_{2i} - 1\right)$.

$$\text{For } n \geq 3,\ \frac{L_{2^n-2}(x)}{L_{2^n-1}(x)} = \left(L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, -\sum_{i=0}^{2^{n-1}-2} L_{2i} + 1 \right).$$

$$\text{For } n \geq 1,\ \frac{L_{3\cdot 2^n-1}(x)}{L_{3\cdot 2^n}(x)} = (L_1, -L_2, L_4, \ldots, L_{2^n}, L_{2^{n+1}} - 1).$$

It is interesting to note the $L_{2^n}/2$ entry, in light of the last convergent of (4).

TABLE 2. Expansions of Quotients of Lucas Polynomials

n	Pierce-Engel Expansion of $L_{n-1}(x)/L_n(x)$
1	$L_1/2$
2	$L_1, -L_2/2$
3	$L_1, -(L_2+1)$
4	$L_1, -L_2, L_4/2$
5	$L_1, -L_2, L_4+L_2+1$
6	$L_1, -L_2, L_4-1$
7	$L_1, -L_2, L_4, -(L_6+L_4+L_2+1)$
8	$L_1, -L_2, L_4, L_8/2$
9	$L_1, -L_2, L_4, L_8+L_6+L_4+L_2+1$
10	$L_1, -L_2, L_4, L_8-L_4+1$
11	$L_1, -L_2, L_4, L_8+L_2, L_{10}+L_8+L_6+L_4+L_2+1$
12	$L_1, -L_2, L_4, L_8-1$
13	$L_1, -L_2, L_4, L_8, -(L_{10}+L_4), -(L_{12}+L_{10}+L_8+L_6+L_4+L_2+1)$
14	$L_1, -L_2, L_4, L_8, L_{12}-L_8+L_4-1$
15	$L_1, -L_2, L_4, L_8, -(L_{14}+L_{12}+L_{10}+L_8+L_6+L_4+L_2+1)$
16	$L_1, -L_2, L_4, L_8, L_{16}/2$
17	$L_1, -L_2, L_4, L_8, L_{16}+L_{14}+L_{12}+L_{10}+L_8+L_6+L_4+L_2+1$
18	$L_1, -L_2, L_4, L_8, L_{16}-L_{12}+L_8-L_4+1$

PIERCE EXPANSIONS OF QUOTIENTS OF FIBONACCI NUMBERS

The limiting value of F_{n-1}/F_n or L_{n-1}/L_n is the same: $(\sqrt{5}-1)/2$. Hence, Engel expansions eventually begin with the pattern of numbers in the Engel expansion of $(\sqrt{5}-1)/2$:

$$2, 5, 6, 13, 16, 16, 38, 48, 58, 104, 177, 263, \ldots,$$

i.e.,

$$\frac{\sqrt{5}-1}{2} = \frac{1}{2} + \frac{1}{2\cdot 5} + \frac{1}{2\cdot 5\cdot 6} + \cdots.$$

There is no pattern apparent in this sequence. In contrast, Pierce expansions begin

$$1, 2, 4, 17, 19, 5777, 5779, 192900153617, \ldots,$$

corresponding to

$$\frac{\sqrt{5}-1}{2} = 1 - \frac{1}{2} + \frac{1}{2\cdot 4} - \frac{1}{2\cdot 4\cdot 17} + \cdots.$$

The Pierce expansion has been analyzed before [13]. it is convenient to express it as

$$\{1, c_0 - 1, c_0 + 1, c_1 - 1, c_1 + 1, \ldots\},$$

where $c_0, c_1, c_2, \ldots = 3, 18, 5778, \ldots$ is the sequence given by the recurrence

$$c_0 = 3,\ \ c_{n+1} = c_n^3 - 3c_n \quad \text{for } n \ge 0.$$

For F_{n-1}/F_n or L_{n-1}/L_n, any particular choice of n gives a rational number and, hence, a finite Pierce expansion, and it often happens that the form of the finite expansion can be given conveniently in terms of the elements of $\{c_i\}$. It turns out to be powers of three that govern the patterns arising, and there are similar results for the Fibonacci and Lucas sequences.

Shallit [13] observed that, for $k \ge 0$,

$$c_k = \left(\frac{3+\sqrt{5}}{2}\right)^{3^k} + \left(\frac{3-\sqrt{5}}{2}\right)^{3^k}.$$

This relates $\{c_k\}$ to the well-known formulas

$$F_n = (\phi^n - \hat{\phi}^n)/\sqrt{5}, \quad L_n = \phi^n + \hat{\phi}^n,$$

where $\phi = (1+\sqrt{5})/2$ and $\hat{\phi} = (1-\sqrt{5})/2$.

Theorem 7: For $k \ge 1$, $F_{3^k-1}/F_{3^k} = (1, c_0 - 1, c_0 + 1, c_1 - 1, c_1 + 1, \ldots, c_{k-1} - 1)$.

We prove this with the aid of several lemmas. The lemmas may be of independent interest for the factorizations they provide for certain Fibonacci and Lucas numbers.

Lemma 8: $c_k = L_{2\cdot 3^k}$, $k \ge 0$.

Proof:

$$c_k = \left(\frac{3+\sqrt{5}}{2}\right)^{3^k} + \left(\frac{3-\sqrt{5}}{2}\right)^{3^k} = (\phi^2)^{3^k} + (\hat{\phi}^2)^{3^k} = L_{2\cdot 3^k}.$$

A similar sequence, introduced by Shallit in [12], provides a formula for the 3^kth Lucas number.

Lemma 9: $F_{3^k} = (c_0 - 1)(c_1 - 1)\cdots(c_{k-1} - 1)$, $k \ge 1$.

Proof: For $k = 1$, $F_3 = 2 = c_0 - 1$. Now, using induction on k,

$$\begin{aligned}
(c_0 - 1)(c_1 - 1)\cdots(c_k - 1) &= F_{3^k}(c_k - 1) \\
&= (\phi^{3^k} - \hat{\phi}^{3^k})(\phi^{2\cdot 3^k} + \hat{\phi}^{2\cdot 3^k} - 1)/\sqrt{5} \quad \text{(by Lemma 8)} \\
&= (\phi^{3^{k+1}} - \hat{\phi}^{3^{k+1}})/\sqrt{5} \quad (\text{since } \phi\hat{\phi} = -1) \\
&= F_{3^{k+1}}.
\end{aligned}$$

Lemma 10: $L_{3^k} = (c_0+1)(c_1+1)\cdots(c_{k-1}+1),\ k \ge 1$.

Proof: Again induct on k.

Lemma 11: $F_{2\cdot 3^k} = (c_0-1)(c_0+1)\cdots(c_{k-1}-1)(c_{k-1}+1),\ k \ge 1$.

Proof: $F_{2\cdot 3^k} = F_{3^k}L_{3^k}$, and the result follows from Lemmas 9 and 10.

Lemma 12: $c_k = L_{3^k}^2 + 2,\ k \ge 0$.

Proof: $L_{3^k}^2 + 2 = (\phi^{3^k} + \hat{\phi}^{3^k})^2 = \phi^{2\cdot 3^k} + \hat{\phi}^{2\cdot 3^k} + 2(\phi\hat{\phi})^{3^k} + 2 = \phi^{2\cdot 3^k} + \hat{\phi}^{2\cdot 3^k} = c_k$.

By Lemma 8, this says $L_{2\cdot 3^k} = L_{3^k}^2 + 2$, so Lemma 12 also follows from the identity $L_{4n-2} = L_{2n-1}^2 + 2,\ n \ge 1$.

Lemma 13: $F_{3^{k+1}-1} = F_{3^k-1}(L_{3^k}^2 + 1) + L_{3^k},\ k \ge 1$.

Proof: The left-hand side may be written as $(\phi^{3^{k+1}-1} - \hat{\phi}^{3^{k+1}-1})/\sqrt{5}$. Write the right-hand side as $(\phi^{3^k-1} - \hat{\phi}^{3^k-1})(\phi^{2\cdot 3^k} + \hat{\phi}^{2\cdot 3^k} - 1)/\sqrt{5} + \phi^{3^k} + \hat{\phi}^{3^k}$ by applying Lemma 12. This may be expanded as

$$(\phi^{3^{k+1}-1} - \hat{\phi}^{3^{k+1}-1} + \phi^{3^k-1}\hat{\phi}^{2\cdot 3^k} - \hat{\phi}^{3^k-1}\phi^{2\cdot 3^k} - \phi^{3^k-1} + \hat{\phi}^{3^k-1} + \sqrt{5}\phi^{3^k} + \sqrt{5}\hat{\phi}^{3^k})/\sqrt{5}$$
$$= F_{3^{k+1}-1} + ((\phi\hat{\phi})^{3^k}(\hat{\phi}^{3^k}\phi^{-1} - \phi^{3^k}\hat{\phi}^{-1}) - \phi^{3^k}\phi^{-1} + \hat{\phi}^{3^k}\hat{\phi}^{-1} + \sqrt{5}\phi^{3^k} + \sqrt{5}\hat{\phi}^{3^k})/\sqrt{5}$$
$$= F_{3^{k+1}-1} + (\hat{\phi}^{3^k}(-\phi^{-1} + \hat{\phi}^{-1} + \sqrt{5}) + \phi^{3^k}(\hat{\phi}^{-1} - \phi^{-1} + \sqrt{5}))/\sqrt{5}.$$

But this is just $F_{3^{k+1}-1}$, since $\hat{\phi}^{-1} - \phi^{-1} + \sqrt{5} = 0$.

Proof of Theorem 7: The proof is by induction on k. For $k=1$, $F_2/F_3 = 1/2 = (1, c_0 - 1)$. Now assume the theorem holds for k, and consider

$$(1, c_0-1, c_0+1, \ldots, c_k-1) = \frac{F_{3^k-1}}{F_{3^k}} + \frac{1}{(c_0-1)(c_o+1)\cdots(c_{k-1}-1)}\left(\frac{1}{c_{k-1}+1} - \frac{1}{(c_{k-1}+1)(c_k-1)}\right)$$

$$= \frac{F_{3^k-1}}{F_{3^k}} + \frac{1}{(c_0-1)(c_0+1)\cdots(c_{k-1}-1)}\,\frac{c_k-2}{(c_{k-1}+1)(c_k-1)}$$

$$= \frac{1}{F_{3^k}}\left(F_{3^k-1} + \frac{c_k-2}{L_{3^k}(c_k-1)}\right) \quad \text{by Lemmas 9 and 10}$$

$$= \frac{F_{3^k-1}L_{3^k}(c_k-1) + c_k - 2}{F_{3^k}L_{3^k}(c_k-1)}$$

$$= \frac{F_{3^k-1}(c_k-1) + (c_k-2)/L_{3^k}}{F_{3^{k+1}}} \quad \text{by Lemma 9}$$

$$= \frac{F_{3^k-1}(L_{3^k}^2+1)+L_{3^k}}{F_{3^{k+1}}} \quad \text{by Lemma 12}$$

$$= \frac{F_{3^{k+1}-1}}{F_{3^{k+1}}} \quad \text{by Lemma 13.}$$

We note that the Pierce expansion considered by Shallit [13] is similar to but not the same as that of Theorem 7 or Theorem 14 below.

Theorem 14: For $k \geq 1$,

$$\frac{L_{3^k-1}}{L_{3^k}} = (1, c_0-1, c_0+1, c_1-1, c_1+1, \ldots, c_{k-2}-1, c_{k-2}+1, c_{k-1}+1).$$

Proof: By Theorem 7,

$$\begin{aligned}
&(1, c_0-1, c_0+1, c_1-1, c_1+1, \ldots, c_{k-2}-1, c_{k-2}+1, c_{k-1}+1) \\
&= \frac{F_{3^k-1}}{F_{3^k}} + \frac{1}{(c_0-1)(c_0+1)\cdots(c_{k-2}+1)}\left(\frac{1}{c_{k-1}-1} - \frac{1}{c_{k-1}+1}\right) \\
&= \frac{F_{3^k-1}}{F_{3^k}} + \frac{2}{F_{3^k}L_{3^k}} = \frac{F_{3^k-1}L_{3^k}+2}{L_{3^k}F_{3^k}} = \frac{L_{3^k-1}}{L_{3^k}}.
\end{aligned}$$

The last step follows because

$$\begin{aligned}
F_{3^k-1}L_{3^k}+2 &= (\phi^{3^k-1}-\hat{\phi}^{3^k-1})(\phi^{3^k}+\hat{\phi}^{3^k})/\sqrt{5}+2 \\
&= (\phi^{2\cdot 3^k-1}-\hat{\phi}^{2\cdot 3^k-1}+\hat{\phi}^{-1}-\phi^{-1}+2\sqrt{5})/\sqrt{5} \\
&= (\phi^{2\cdot 3^k-1}-\hat{\phi}^{2\cdot 3^k-1}+\phi^{-1}-\hat{\phi}^{-1})/\sqrt{5} \\
&= (\phi^{3^k-1}+\hat{\phi}^{3^k-1})(\phi^{3^k}-\hat{\phi}^{3^k})/\sqrt{5} = L_{3^k-1}F_{3^k}.
\end{aligned}$$

There are many related identities that can be noted. We close with the omnibus theorem below, indicating several patterns that we have observed. The proofs are omitted, since the identities may be derived in the same way as the paradigms in Theorems 7 and 14.

Theorem 15: For $n = 2\cdot 3^k$, $k \geq 1$,

$$\frac{F_{n-1}}{F_n} = (1, c_0-1, c_0+1, \ldots, c_{k-1}-1, c_{k-1}+1).$$

For $n = 4\cdot 3^k$, $k \geq 1$,

$$\frac{F_{n-1}}{F_n} = (1, c_0-1, c_0+1, \ldots, c_{k-1}-1, c_{k-1}+1, c_k).$$

For $n = 8\cdot 3^k$, $k \geq 1$,

$$\frac{F_{n-1}}{F_n} = (1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, c_k+c_{k+1}).$$

For $n=5\cdot 3^k$, $k\geq 1$,

$$\frac{F_{n-1}}{F_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, (c_k-1)c_k-1).$$

For $n=7\cdot 3^k$, $k\geq 1$,

$$\frac{F_{n-1}}{F_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, ((c_k-1)c_k-1)c_k-(c_k-1)).$$

For $n=2\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_{k-1}-1, c_{k-1}+1, c_k/2).$$

For $n=4\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+2, c_k^2/2-1).$$

For $n=8\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, c_{k+1}-c_k, c_kc_{k+1}/2-(c_k^2/2-1)).$$

For $n=5\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k, c_k+2, c_k^2+c_k-1).$$

For $n=7\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, (c_k^2+c_k-2)c_k-1).$$

We note finally that nonlinear recurrence relations also arise in the expansions of certain rational numbers by means of other related algorithms (see [5], [15]).

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NEW EDITORIAL POLICIES

The Board of Directors of The Fibonacci Association during their last business meeting voted to incorporate the following two editorial policies effective January 1, 1995

1. All articles submitted for publication in *The Fibonacci Quarterly* will be blind refereed.
2. In place of Assistant Editors, *The Fibonacci Quarterly* will change to utilization of an Editorial Board.