A GENERALIZATION OF A RESULT OF D'OCAGNE

R. S. Melham and A. G. Shannon
University of Technology, Sydney 2007, Australia
(*Submitted July 1993*)

1. INTRODUCTION

In this paper we consider some aspects of sequences generated by the m^{th} order homogeneous linear recurrence relation

$$R_n = \sum_{i=1}^{m} a_i R_{n-i} \quad \text{for } m \geq 2, \tag{1.1}$$

where $a_m \neq 0$ and the underlying field is the complex numbers. To generate a sequence $\{R_n\}_{n=0}^{\infty}$, we specify initial values $R_0, R_1, \ldots, R_{m-1}$. Indeed, this sequence can be extended to negative subscripts by using (1.1), and with this convention we simply write $\{R_n\}$.

For the case $m = 2$, we adopt the notation of Hordam [3] and write

$$W_n = W_n(a, b; p, q), \tag{1.2}$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a,\ W_1 = b. \tag{1.3}$$

If $(R_0, \ldots, R_{m-2}, R_{m-1}) = (0, \ldots, 0, 1)$, we write $\{R_n\} = \{U_n\}$. The sequence $\{U_n\}$ is called the fundamental sequence generated by (1.1). It is "fundamental" in the sense that, if $\{R_n\}$ is any sequence generated by (1.1), then there exist complex numbers $b_0, \ldots, b_{m-1}$ depending upon $a_1, \ldots, a_m$ and $R_0, \ldots, R_{m-1}$ such that

$$R_n = \sum_{i=0}^{m-1} b_i U_{n+i} \quad \text{for all integers } n. \tag{1.4}$$

In this regard, see Jarden [4], p. 114 or Dickson [1], p. 409, where this result is attributed to D'Ocagne. In §2 we generalize this idea.

For the Fibonacci and Lucas numbers, it can be proved that

$$L_n^2 + L_{n+1}^2 = 5(F_n^2 + F_{n+1}^2). \tag{1.5}$$

More generally, for the second-order fundamental and primordial sequences of Lucas [5] defined by

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q), \end{cases} \tag{1.6}$$

where $\Delta = p^2 - 4q \neq 0$, we have

$$-qV_n^2 + V_{n+1}^2 = \Delta(-qU_n^2 + U_{n+1}^2). \tag{1.7}$$

In §3 we demonstrate the existence of a result analogous to (1.7) for any two sequences generated by (1.1).

2. A GENERALIZATION OF D'OCAGNE'S RESULT

Let $\{R_n\}$ and $\{S_n\}$ be any two sequences generated by (1.1). Define the $(m+1)\times(m+1)$ determinant D_n, for all integers n, by

$$D_n = \begin{vmatrix} R_n & S_n & S_{n+1} & \cdots & S_{n+m-1} \\ R_{m-1} & S_{m-1} & S_m & \cdots & S_{2m-2} \\ R_{m-2} & S_{m-2} & S_{m-1} & \cdots & S_{2m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_0 & S_0 & S_1 & \cdots & S_{m-1} \end{vmatrix}.$$

Theorem 1: $D_n = 0$ for all integers n.

Proof: $D_0 = D_1 = \cdots = D_{m-1} = 0$ since, in each case, we have an $(m+1)\times(m+1)$ determinant with two identical rows. Now expanding D_n along the top row, we see that D_n is a linear combination of $R_n, S_n, \ldots, S_{n+m-1}$. Therefore, since each of the sequences $\{R_n\}, \{S_n\}, \ldots, \{S_{n+m-1}\}$ is generated by (1.1) then so is $\{D_n\}$. But $\{D_n\}$ has m successive terms that are zero and so all its terms are zero. This completes the proof. □

We now come to the main result of this section.

Corollary 1: There exist constants c and c_{oj}, $0 \le j \le m-1$, such that

$$cR_n = \sum_{j=0}^{m-1} c_{oj} S_{n+j} \quad \text{for all integers } n. \tag{2.1}$$

Proof: Expand D_n along the top row. □

Equation (2.1) generalizes D'Ocagne's result (1.4), where the b_i are normally specified without the use of determinants. If $\{S_n\} = \{U_n\}$, then c, which is the minor of R_n is unity and we obtain an equivalent form of D'Ocagne's result.

3. A RESULT CONCERNING SUMS OF SQUARES

From (2.1) we have, for any integer i,

$$cR_{n+i} = \sum_{j=0}^{m-1} c_{oj} S_{n+i+j}. \tag{3.1}$$

Using (1.1), the right side of (3.1) can be written in terms of $S_n, S_{n+1}, \ldots, S_{n+m-1}$. That is, for any integer i there exist constants c_{ij}, $0 \le j \le m-1$, such that

$$cR_{n+i} = \sum_{j=0}^{m-1} c_{ij} S_{n+j}. \tag{3.2}$$

Write $\ell = \binom{m}{2}$. Then, for parameters $d_0, d_1, \ldots, d_\ell$ we have, from (3.2),

$$c^2 \sum_{i=0}^{\ell} d_i R_{n+i}^2 = \sum_{j=0}^{m-1} S_{n+j}^2 \sum_{i=0}^{\ell} d_i c_{ij}^2 + 2 \sum_{0 \le j < k \le m-1} S_{n+j} S_{n+k} \sum_{i=0}^{\ell} d_i c_{ij} c_{ik}. \tag{3.3}$$

Consider the system of equations

$$\sum_{i=0}^{\ell} d_i c_{ij} c_{ik} = 0, \quad 0 \le j < k \le m-1, \tag{3.4}$$

in the unknowns $d_0, d_1, \ldots, d_\ell$. Since (3.4) is a system of ℓ homogeneous linear equations in $\ell+1$ unknowns, there are an infinite number of solutions $(d_0, d_1, \ldots, d_\ell)$. Choose any nontrivial solution and put

$$e_i = c^2 d_i, \qquad 0 \le i \le \ell,$$

$$f_j = \sum_{i=0}^{\ell} d_i c_{ij}^2, \quad 0 \le j \le m-1.$$

Making these substitutions in (3.3), we have succeeded in proving the following theorem.

Theorem 2: Let $\{R_n\}$ and $\{S_n\}$ be any two sequences generated by the recurrence (1.1). Then there exist constants e_i, $0 \le i \le \ell = \binom{m}{2}$, and f_i, $0 \le i \le m-1$, not all zero such that, for all integers n,

$$\sum_{i=0}^{\ell} e_i R_{n+i}^2 = \sum_{i=0}^{m-1} f_i S_{n+i}^2. \tag{3.5}$$

Theorem 2 shows the existence of a result analogous to (1.7) for any two sequences generated by (1.1).

Example 1: Let $\{W_n\}$ and $\{S_n\}$ be any two sequences generated by the recurrence (1.3). Then, after some tedious algebra, we obtain the following determinantal identity:

$$\begin{vmatrix} \begin{vmatrix} S_n^2 & S_{n+1}^2 \\ \begin{vmatrix} W_2 & S_1 \\ W_1 & S_0 \end{vmatrix} & \begin{vmatrix} S_2 & W_1 \\ S_3 & W_2 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} W_n^2 & W_{n+1}^2 \\ \begin{vmatrix} S_2 & W_1 \\ S_3 & W_2 \end{vmatrix} & q^2\begin{vmatrix} W_2 & S_1 \\ W_1 & S_0 \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} S_1 & S_2 \\ S_2 & S_3 \end{vmatrix} & -q\begin{vmatrix} W_1 & W_2 \\ W_2 & W_3 \end{vmatrix} \end{vmatrix} = 0. \tag{3.6}$$

Example 2: For a fixed integer k, consider the sequences $\{F_{kn}\}$ and $\{L_{kn}\}$. They both satisfy the recurrence (1.3) with $p = L_k$ and $q = (-1)^k$. Substitution into (3.6) yields

$$5(F_{kn}^2 + (-1)^{k-1} F_{k(n+1)}^2) = L_{kn}^2 + (-1)^{k-1} L_{k(n+1)}^2. \tag{3.7}$$

Example 3: In (1.1), taking $m = 3$ and $a_1 = a_2 = a_3 = 1$, we have

$$R_n = R_{n-1} + R_{n-2} + R_{n-3}. \tag{3.8}$$

Feinberg [2] referred to sequences generated by (3.8) as Tribonacci sequences.

For $(R_0, R_1, R_2) = (0, 0, 1)$ write $\{R_n\} = \{U_n\}$.

For $(R_0, R_1, R_2) = (3, 1, 3)$ write $\{R_n\} = \{V_n\}$.

Then $\{V_n\}$ bears the same relation to $\{U_n\}$ as does the Lucas sequence to the Fibonacci sequence (see [6], p. 300).

Now assuming a relationship between $\{U_n\}$ and $\{V_n\}$ of the form (3.5) and solving for the coefficients e_i and f_i yields

$$34V_n^2 - 30V_{n+1}^2 + V_{n+2}^2 + 9V_{n+3}^2 = -154U_n^2 + 176U_{n+1}^2 + 726U_{n+2}^2. \tag{3.9}$$

Alternatively, we have

$$46U_n^2 - 50U_{n+1}^2 - 114U_{n+2}^2 + 54U_{n+3}^2 = -7V_n^2 + 12V_{n+1}^2 - V_{n+2}^2. \tag{3.10}$$

4. OPEN QUESTION

Is there a result analogous to (3.5) for higher powers?

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AMS Classification Numbers: 11B37, 11B39

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