ON THE SYSTEM OF CONGRUENCES $\prod_{j\neq i} n_j \equiv 1 \bmod n_i$

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We seek integers $n_1, \ldots, n_k$, all ≥ 2, for which

$$\prod_{j\neq i} n_j \equiv 1 \bmod n_i \tag{1}$$

for all i. Problems of this sort arise, for instance, in connection with the Chinese remainder theorem and structure theory for finite Abelian groups. Curiously, this system has received little attention compared to the system

$$\prod_{j\neq i} n_j \equiv -1 \bmod n_i \tag{2}$$

(see [3], [5], [6], [7], [11]). System (2) has attracted interest because it is equivalent to the unit fraction equation

$$\sum_{i=1}^{k} 1/n_i + 1/\prod_{i=1}^{k} n_i = m, \text{ an integer.} \tag{3}$$

Especially for $m = 1$ this problem is not only interesting in its own right in the field of Egyptian fractions, but also has proved to have application to the topology of singular points of algebraic surfaces [4]. In this paper we will apply what is known about system (2) to derive a large number of solutions to system (1). All solutions to (1) with 7 or fewer terms are given in the appendices, together with techniques for producing some 398 solutions with 8 terms and 1411 with 9 terms.

Lemma 1: Let $n_1, \ldots, n_k$ be positive integers, relatively prime in pairs. Put

$$X = \prod_{i=1}^{k} n_i, \qquad Y = \sum_{i=1}^{k} \prod_{j\neq i} n_j,$$

and let D be the smallest positive residue of $-Y \bmod X$.

(a) If $X \equiv 1$ (resp. -1) mod D, then $n_1, \ldots, n_k, n_{k+1}$ satisfy (1) [resp. (2)] for $n_{k+1} = (X-1)/D$ [resp. $(X+1)/D$].

(b) If $X^2 - D$ admits a factor $P \equiv -X \bmod D$, then $n_1, \ldots, n_k, n_{k+1}, n_{k+2}$ satisfy (1) for $n_{k+1} = (X+P)/D$ and $n_{k+2} = (X+Q)/D$, where $Q = (X^2 - D)/P$.

Proof: For example, see [4], Proposition 12. (a) is immediate. For (b) we have

(i) $(\prod_{i=1}^{k} n_i)n_{k+1} = Pn_{k+2} + 1$,

(ii) $(\prod_{i=1}^{k} n_i)n_{k+2} = Qn_{k+1} + 1$,

while for $i \leq k$, computing modulo n_i gives

(iii) $(\prod_{j\neq i} n_j)n_{k+1}n_{k+2} \equiv YPQD^{-2} \equiv (-D)(-D)D^{-2} \equiv 1$,

where D^{-1} is well defined mod n_i since D and X are relatively prime.

As a special case, if $n_1, \ldots, n_k$ satisfy (2), then $D = 1$. Thus,

Corollary 2: Let $n_1, \ldots, n_k$ satisfy (2). Then

(a) $n_1, \ldots, n_k, n_{k+1}$ also satisfy (2) for $n_{k+1} = \prod_{i=1}^k n_i + 1$,

(b) $n_1, \ldots, n_k, n_{k+1}$ satisfy (1) for $n_{k+1} = \prod_{i=1}^k n_i - 1$, and

(c) if $P \big| \prod_{i=1}^k n_i^2 - 1$, then $n_1, \ldots, n_k, n_{k+1}, n_{k+2}$ satisfy (1) for $n_{k+1} = \prod_{i=1}^k n_i + P$, $n_{k+2} = \prod_{i=1}^k n_i + Q$, where $Q = (\prod_{i=1}^k n_i^2 - 1)/P$.

Since all solutions to (1) are known for $k \le 7$ (see [4]), as well as some 500 independent infinite sequences of solutions for increasingly large k (see [1]), part (b) gives a rich family of solutions to the congruences (1) obtained in this trivial way. To make use of part (c), we must be able to find factors of numbers of the form $\prod_{i=1}^k n_i^2 - 1$. Immediately we have the factors $\prod_{i=1}^k n_i - 1$ and $\prod_{i=1}^k n_i + 1$; hence, the following corollary.

Corollary 3: Let $n_1, \ldots, n_k$ satisfy (2). Then $n_1, \ldots, n_k, n_{k+1}, n_{k+2}$ satisfy (1) for $n_{k+1} = 2\prod_{i=1}^k n_i - 1$, $n_{k+2} = 2\prod_{i=1}^k n_i + 1$ (as well as for $n_{k+1} = \prod_{i=1}^k n_i + 1$, $n_{k+2} = \prod_{i=1}^k n_i^2 + \prod_{i=1}^k n_i - 1$).

By finding further factors of $\prod_{i=1}^k n_i - 1$ and $\prod_{i=1}^k n_i + 1$ for fixed $n_1, \ldots, n_k$ satisfying (2), we can find further solutions to (1) (see Appendix 2 below). But a more fruitful approach has proven to be as follows (cf. [12]). Choose a prime P, then try to find a solution $n_1, \ldots, n_k$ to (2) for which P divides $\prod_{i=1}^k n_i - 1$ or $\prod_{i=1}^k n_i + 1$.

For P a positive integer, consider the relation "succeeds mod P" defined on the set Z_p of integers mod P by y succeeds x mod P if $y = x^2 + x$. We will write $x \prec y$ if there is a finite sequence $x_0 = x, x_1, \ldots, x_\ell = y$, $\ell \ge 1$, such that x_i succeeds x_{i-1} for $i = 1, \ldots, \ell$ ($x \prec x$ is permissible), and we will write $x \preccurlyeq y$ if $x \prec y$ or $x = y$. Some properties of this relation are worked out in [1] in connection with equation (3). To give a particular example, which will be referred to later, for $P = 19$ the relation "succeeds" is represented by the following directed graph.

$$\begin{tikzcd}
\mathbf{9} \arrow[d] & & \mathbf{17} \arrow[d] & \mathbf{16} \arrow[d] & \mathbf{8} \arrow[r] & \mathbf{15} \arrow[d] & \mathbf{10} \arrow[l] & & \mathbf{13} \arrow[d] & \mathbf{7} \arrow[d] & \\
\mathbf{14} \arrow[r] & \mathbf{1} \arrow[r] & \mathbf{2} \arrow[r] & \mathbf{6} \arrow[r] & \mathbf{4} \arrow[lll, bend left] & \mathbf{12} \arrow[l] & \mathbf{3} \arrow[l] & \mathbf{5} \arrow[r] & \mathbf{11} \arrow[r] & \mathbf{-1} \arrow[r] & \mathbf{0} \arrow[loop right]
\end{tikzcd}$$

Proposition 4: Let $n_1, \ldots, n_k$ satisfy (2), let P be a positive integer, and suppose that $\prod_{i=1}^k n_i \preccurlyeq \pm 1$ mod P. Put $n_{k+1} = \prod_{i=1}^k n_i + 1$, and for $\ell = 2, 3, \ldots$, put $n_{k+\ell} = n_{k+\ell-1}^2 - n_{k+\ell-1} + 1$. Then, for some $\ell \ge 1$, $n_1, \ldots, n_{k+\ell-1}, n_{k+\ell} + P - 1, n_{k+\ell} + Q - 1$ satisfy (1), for appropriate choice of Q.

Proof: First we note that $\forall \ell\; n_{k+\ell} = \prod_{i<k+\ell} n_i + 1$. Thus $n_1, \ldots, n_{k+\ell}$ satisfy (2) $\forall \ell$. Furthermore, the products $\prod_{i\le k+\ell} n_i = n_{k+\ell+1} - 1$ satisfy the relation

$$\prod_{i\le k+\ell} n_i = \left(\prod_{i\le k+\ell-1} n_i\right)\left(\prod_{i\le k+\ell-1} n_i + 1\right),$$

that is, $\prod_{i\le k+\ell} n_i$ succeeds $\prod_{i\le k+\ell-1} n_i$ mod P. Since $\prod_{i=1}^k n_i \preccurlyeq \pm 1$, it follows that P divides

$\prod_{i \leq k+\ell}^{k} n_i \mp 1$ for some ℓ. By Lemma 1(c), then, $n_1, \ldots, n_{k+\ell-1}, n_{k+\ell} + P - 1$, $n_{k+\ell} + Q - 1$ satisfy (1) for this choice of ℓ and for $Q = ((n_{k+\ell} - 1)^2 - 1) / P$.

Remark: For a few small primes P, $x \preccurlyeq \pm 1 \bmod P$ for **every** integer $x \bmod P$ except $x = 0$. $P = 2, 3, 5, 7$, and 19 (see graph above), for instance, have this property. Thus, we have

Corollary 5: Let $P = 2, 3, 5, 7$, or 19. Let $n_1, \ldots, n_k$ satisfy (2), where $P \nmid n_i \forall i$. Then $\prod_{i=1}^{k} n_i \preccurlyeq \pm 1$ and we obtain a solution to (1) as in Proposition 4.

Note: In connection with the prime $P = 2$, it should be mentioned that no solution to (1) or (2) is known with each n_i odd. For $P = 3$, the shortest solution to (2) with each $n_i \not\equiv 0 \bmod 3$ is (2, 5, 7, 11, 17, 157, 961, 4398619). This leads to the solution (2, 5, 7, 11, 17, 157, 961, 4398619, 8687184244716671, 75467170101653548887992820605569) to (1), where no term is divisible by 3. Indeed, applying Corollary 2(c) to appropriate factors of

$$(2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 157 \cdot 961 \cdot 4398619)^2 - 1 = 3 \cdot 719 \cdot 2287 \cdot 466201 \cdot 2715929 \cdot 12082314665809$$

gives sixteen distinct solutions to (1) with 10 terms, none $\equiv 0 \bmod 3$. However, there may be a shorter solution to (1) with this feature.

We also observe that for $P = 5$ and $P = 19$, $1 \prec 1$. Thus, $P \mid \prod_{i=1}^{k+\ell} n_i - 1$ for infinitely many ℓ, and we have an infinite sequence of solutions to (1) based on these primes. In general,

Corollary 6: Let $n_1, \ldots, n_k$ satisfy (2) and let P be an integer such that $\prod_{i=1}^{k} n_i \preccurlyeq 1$ and $1 \prec 1$. Then the procedure of Proposition 4 gives infinitely many solutions to (1).

Proof: Let ℓ_0 be the smallest of the indices for which $\prod_{i=1}^{k+\ell} n_i \equiv 1 \bmod P$, and let m_0 be the smallest positive integer for which we have a chain of successors $1 \to x_i \to x_2 \to \cdots \to x_{m_0-1} \to 1 \bmod P$. Then $\prod_{i=1}^{k+\ell_0+jm_0} n_i \equiv 1 \bmod P \; \forall j = 1, 2, \ldots$, each of which gives a solution to (1) by Lemma 2(c).

Primes $P < 1000$ for which $1 \prec 1$ are 5, 19, 31, 41, 89, 409, 431, 461, 569, and 661.

PRIMALITY TESTING AND FIBONACCI NUMBERS

The methods of the previous section show that when $\prod_{i=1}^{k} n_i \pm 1$ have many factors for various solutions $n_1, \ldots, n_k$ to (2), then we obtain many solutions to (1). It is equally interesting to inquire whether these numbers are prime. For instance, $2 \cdot 3 \pm 1 = \{5, 7\}$, $2 \cdot 3 \cdot 7 \pm 1 = \{41, 43\}$, $2 \cdot 3 \cdot 7 \cdot 43 \cdot 1807 \pm 1 = \{3263441, 3263443\}$, and $2 \cdot 3 \cdot 11 \cdot 23 \cdot 31 \pm 1 = \{47057, 47059\}$ are four pairs of twin primes, where the indicated factors are solutions to (2). In the case of $N = \prod n_i + 1$, primality tests of Fermat type are especially appropriate because we know many factors of $N - 1$. Indeed, if there is an integer y for which $y^{N-1} \equiv 1 \bmod N$ but $y^{\prod_{j \neq i} n_j} \not\equiv 1 \bmod N \; \forall i$, then N is "very probably prime" and we need only find the factors of each n_i to complete the test. Some solutions to (2) for which $\prod_{i=1}^{k} n_i + 1$ is prime are (2), (2, 3), (2, 3, 7), (2, 3, 11, 23, 31), (2, 3, 7, 43, 1807), (2, 3, 7, 47, 395), (2, 3, 7, 47, 403, 19403), (2, 3, 7, 47, 415, 8111), (2, 3, 7, 55, 179, 24323), (2, 3, 7, 43, 3263, 4051, 2558951), (2, 3, 7, 55, 179, 24323, 10057317271), (2, 3, 11, 23, 31, 47423, 6114059), and (2, 3, 11, 25, 29, 1097, 2753). These are all such examples with $k \leq 7$.

For $\Pi n_i - 1$ we will focus our attention on the sequence $2, 3, 7, 43, \ldots, y_k, \ldots$, where $y_k = \Pi_{i<k} y_i + 1$. By Corollary 2(a), $\forall k$ the first k terms of this sequence satisfy (2). Put $x_k = \Pi_{i\le k} y_i = y_{k+1} - 1$. Then $x_k = x_{k-1}^2 + x_{k-1}$ and we have the succession relation $1 \to 2 \to 6 \to \cdots \to x_{k-1} \to 1$ mod P for any divisor P of $x_k - 1$.

Lemma 7:

(a) If $m|k$ then $(x_m - 1)|(x_k - 1)$.

(b) (i) $(x_{k-1}+2)|(x_k - 2)$ and (ii) if $\ell|(k-1)$ then $(x_\ell - 1)|(x_k - 2)$.

Proof:

(a) If $m|k$, say $k = md$. Then mod $(x_m - 1)$ we have the sequence of successions $1 \to 2 \to 6 \to \cdots \to x_{m-1} \to 1$, and after d repetitions of this loop we obtain $x_k \equiv 1$ mod $(x_m - 1)$ and $(x_m - 1)|(x_k - 1)$.

(b) From $x_k = x_{k-1}^2 + x_{k-1}$, we have $x_k - 2 = (x_{k-1}+2)(x_{k-1}-1)$, hence assertion (i). Now assertion (ii) follows from (a).

Corollary 8 [of (a)]: If k is composite, then so is $x_k - 1$.

If k is prime, then $x_k - 1$ may be prime and, again, since we know several factors of $x_k - 2$ by (b) above, primality tests of Fermat type are available. A variation on this theme is to apply a Lucas-type test using the Fibonacci numbers. As a historical sidelight, in connection with the unit fraction equation (3), Fibonacci was the first to prove, in 1202, that if $m, n_1, \ldots, n_k$ is any collection of positive integers with $\sum_{i=1}^{k} 1/n_i < m$, then there exist $\ell, n_{k+1}, \ldots, n_{k+\ell}$ such that $\sum_{i=1}^{k+\ell} 1/n_i = m$ (but not necessarily with $n_{k+\ell} = \Pi_{i<k+\ell} n_i$).

Lemma 9: Let $\{x_\ell\}$ denote the sequence of positive integers defined by $x_0 = 1$, $x_\ell = x_{\ell-1}^2 + x_{\ell-1}$ for $\ell \ge 1$, and let k be an odd prime. Put $y = x_{k-1} + 1$. Then $\forall i = 1, 2, \ldots$,

$$y^i \equiv F_i y + F_{i-1} \bmod (x_k - 1), \tag{4}$$

where $\{F_i\}$ denotes the Fibonacci numbers, beginning with $F_0 = 0, F_1 = 1$. Furthermore, both y and $2y-1$ are invertible in the ring $Z_{(x_k-1)}$ of integers mod $(x_k - 1)$ and $\forall i$

$$F_i \equiv (y^i - (-y)^{-i})(2y-1)^{-1} \bmod (x_k - 1). \tag{5}$$

Proof: For the first assertion we use induction on i. If $i = 1$ the claim is just that

$$y \equiv F_1 y + F_0 = 1y + 0.$$

Now let $i > 1$ and assume the claim to be true for all smaller indices—in particular, that $y^{i-1} \equiv F_{i-1}y + F_{i-2}$ mod $(x_k - 1)$. From $y = x_{k-1} + 1$ and $x_{k-1}^2 + x_{k-1} = x_k$ we have

$$y^2 = x_{k-1}^2 + x_{k-1} + x_{k-1} + 1 = x_k + y \equiv y + 1 \bmod (x_k - 1). \tag{6}$$

Thus, modulo $(x_k - 1)$,

$$\begin{aligned} y^i = y(y^{i-1}) &\equiv y(F_{i-1}y + F_{i-2}) \equiv F_{i-1}y^2 + F_{i-2}y \\ &\equiv F_{i-1}(y+1) + F_{i-2}y \equiv (F_{i-1} + F_{i-2})y + F_{i-1} \equiv F_i y + F_{i-1} \end{aligned}$$

as required.

As for invertibility of y, note that

$$y x_{k-1} = (x_{i-1}+1)x_{k-1} = x_{k-1}^2 + x_{k-1} = x_k \equiv 1 \bmod (x_k - 1),$$

so that y^{-i} exists in $Z_{(x_k-1)}$ and is equal to x_{k-1}. Furthermore, we have

$$(-y^{-1})^2 = x_{k-1}^2 \equiv -x_{k-1} + 1 = (-y^{-1}) + 1 \bmod (x_k - 1).$$

Since this is equation (6) above with $-y^{-1}$ in place of y, the same inductive proof as above shows that also

$$(-y)^{-i} \equiv F_i(-y^{-1}) + F_{i-1} \bmod (x_k - 1). \tag{7}$$

Subtracting (7) from (6) now gives

$$y^i - (-y)^{-i} \equiv F_i(y + y^{-1}) \equiv F_i(2y-1) \bmod (x_k - 1).$$

To complete the proof of (5), we must show that $2y-1$ is invertible in $Z_{(x_k-1)}$—that is, that $2y-1$ and $x_k - 1$ have no common factors.

To see this, we compute

$$\begin{aligned}(2y-1)^2 &= (2x_{k-1}+1)^2 = 4x_{k-1}^2 + 4x_{k-1} + 1\\ &= 4x_k + 1 = 4(x_k - 1) + 5,\end{aligned}$$

so any common divisor of $2y-1$ and $x_k - 1$ must also divide 5. But in the sequence $\{x_\ell\}_{\ell=0}^{\infty} = \{1, 2, 6, 42, 1806, \ldots\}$, $x_\ell \equiv 2 \bmod 5$ for all odd ℓ. In particular, $x_k - 1 \equiv 1 \bmod 5$, so 5 does not divide $x_k - 1$ and we conclude that $2y-1$ and $x_k - 1$ are mutually prime as claimed. Thus, $2y-1$ is invertible mod $(x_k - 1)$ and the proof of equation (5) is complete.

Remark: Another way to view this connection between the Fibonacci numbers and the powers of y is to note that y and $(-y^{-1})$ are two solutions modulo$(x_k - 1)$ to the quadratic equation $Y^2 - Y - 1 = 0$. That is, we may regard y as the "golden mean" $y = (1+\sqrt{5})/2$ in $Z_{(x_k-1)}$, where 2 is invertible since $(x_k - 1)$ is odd and where $\sqrt{5}$ exists by quadratic reciprocity. Thus, equation (5) is the equivalent in $Z_{(x_k-1)}$ of the well-studied computational formula

$$F_i = \left[\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right] \Big/ \sqrt{5}.$$

Proposition 10: Let $\{x_\ell\}, k, Y$ be as in Lemma 9. Then the sequence of Fibonacci numbers modulo $(x_k - 1)$ repeats with some period λ, where λ divides the order of the multiplicative group $Z_{(x_k-1)}^*$ of invertible elements of $Z_{(x_k-1)}$. If $\lambda = x_k - 2$, then $x_k - 1$ is prime and $Z_{(x_k-1)}^*$ is the cyclic group generated by y.

Proof: In any case, since there are only $(x_k - 1)^2$ pairs of integers mod $(x_k - 1)$, the sequence $\{F_i\}$ in $Z_{(x_k-1)}$ must repeat after at most $(x_k - 1)^2$ terms. Let λ be the smallest positive integer for which $F_{i+\lambda} \equiv F_i$ for all i. By equation (4) of Lemma 9, then, $y^{i+\lambda} \equiv y^i \; \forall i$.

Conversely, if μ is the order of y in the group $Z_{(x_k-1)}^*$, then equation (5) of Lemma 9 shows that

$$F_{i+\mu} \equiv F_i \bmod (x_k - 1) \text{ for all } i.$$

We conclude that $\mu = \lambda$ and that the period of $\{F_i\}$ is the same as the multiplicative order of y in $Z^*_{(x_k-1)}$. Since this order must divide the order of $Z^*_{(x_k-1)}$ by Lagrange's theorem, we have proved the first assertion.

Finally, if $\lambda = x_k - 2$, then $y, y^2, \ldots, y^{x_k-2} = 1$ are all distinct in $Z^*_{(x_k-1)}$, so $|Z^*_{(x_k-1)}| = x_i - 2$ and $x_k - 1$ is coprime to each of $1, 2, \ldots, x_k - 2$. Thus, $x_k - 1$ is prime as claimed, with $Z^*_{(x_k-1)}$ the cyclic group consisting of powers of y.

Remarks: As the proof shows, the condition $\lambda = x_k - 2$ is equivalent to $F_{x_k-2} \equiv 0$ and $F_{x_k-1} \equiv 1 \bmod (x_k - 1)$, but $(F_i, F_{i+1}) \not\equiv (0, 1) \bmod (x_k - 1)\ \forall$ proper divisors i of $x_k - 2$. An example where these computations can be carried out by hand is $k = 3$, $x_k - 1 = 2\cdot 3\cdot 7 - 1 = 41$, $y = 7$. The Fibonacci numbers $(F_{40}, F_{41}) \equiv (0, 1)$ but (F_8, F_9) and $(F_{20}, F_{21}) \not\equiv (0, 1) \bmod 41$, so Z^*_{41} consists of powers of 7. Similarly, $y = 1807$ generates the multiplicative group of integers modulo the prime $x_5 - 1 = 3263441$.

APPLICATION TO ALGEBRAIC SURFACES

Our interest was first attracted to number theoretic problems of this type because of the following considerations from the topology of complex surfaces. Let S be an algebraic surface over C with a normal isolated singular point P. Let $f : \widetilde{S} \to S$ be the minimal normal resolution of singularities with exceptional curve $C = f^{-1}(P) = \bigcup_{i=1}^n C_i$, where each C_i is nonsingular and meets C_j, if at all, transversally in a single point $\forall j \neq i$. C is represented by its dual weighted intersection graph Γ, in which each vertex v_i corresponds to a component C_i, with edges $\{v_i, v_j\}$ whenever C_i meets C_j, and with positive integer weight $w_i = -C_i^2$ assigned to the vertex v_i, where C_i^2 is the self-intersection number (the Chern class of the normal line bundle of the embedding of C_i in $\widetilde{S}$). If each C_i is rational, then Γ completely determines the topology of a neighborhood U of the singular point in S. In particular, if Γ has no cycles then U is the cone on a smooth real three-manifold M whose fundamental group π_1 is generated by $v_1, \ldots, v_n$ with relations $\prod_{j=1}^n v_j^{-(C_i \cdot C_j)} = 1\ \forall i$ and $v_i v_j = v_j v_i$ if C_i meets C_j [9]. From this, it follows that the first homology group of M is the Abelian group with these generators and relations, with order the determinant of the weighted intersection matrix of Γ, written $|\Gamma|$.

This determinant, in turn, can be calculated very quickly using techniques of graph theory in linear algebra [8]. In particular, if Γ is any weighted tree, v_0 a vertex of Γ of weight w_0, we have the following "expansion by a vertex" formula ([2], eq. 2.13). Let $v_1, \ldots, v_k$ be the vertices of Γ that meet v_0, denote by Γ_i the component of $\Gamma - \{v_0\}$ which contains v_i, and put $\Gamma_i' = \Gamma_i - \{v_i\}$. Then

$$|\Gamma| = w_0 \prod_{i=1}^{k} |\Gamma_i| - \sum_{i=1}^{k} |\Gamma_i'| \prod_{j\neq i} |\Gamma_j|. \tag{8}$$

A recurring problem in two-dimensional singularity theory is to classify or to find examples of complex surface singularities whose local fundamental group π_1 satisfies some standard group theoretic criterion, such as being solvable [13] or nilpotent [10]. By the preceding discussion, π_1 is **perfect** (generated by commutators) if and only if Γ is acyclic, each exceptional component C_i is rational, and $|\Gamma| = 1$. The results of this paper give a large family of such "perfect" singularities.

A weighted graph Γ will be called **standard star-shaped** if it consists of linear arms $\Gamma_1, \ldots, \Gamma_k$, each vertex having weight 2, joined at a terminal vertex v_{i1} to a common central vertex v_0 of weight w_0 (see Figure 1).

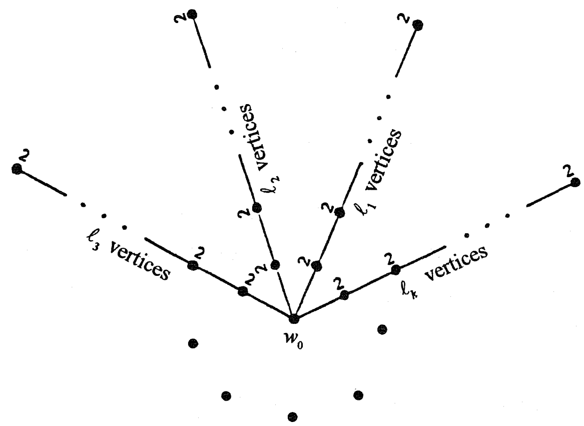

FIGURE 1

Theorem 11: Let $P \in S$ be an isolated complex surface singularity with minimal normal resolution $f: \widetilde{S} \to S$. Suppose that each component of the exceptional curve is rational and that the weighted dual intersection graph Γ of $f^{-1}(P)$ is standard star-shaped with k arms as pictured in Figure 1. For $i = 1, \ldots, k$ put $n_i = \ell_i + 1$, where ℓ_i is the length of the i^{th} arm of Γ. Then the local fundamental group π_1 of P in S is perfect if and only if $n_1, \ldots, n_k$ satisfy the system of congruences (1) with $(\sum_{j=1}^k \prod_{j\neq i} n_j - 1)/\prod_{i=1}^k n_i = k - w_0$.

Proof: The linear graph A_ℓ on ℓ vertices with all weights 2 has determinant $\ell + 1$. Hence, for the graph Γ of Figure 1, formula (8) above becomes

$$|\Gamma| = w_0 \prod_{i=1}^k n_i - \sum_{i=1}^k (n_i - 1) \prod_{j\neq i} n_j = (w_0 - k) \prod_{i=1}^k n_i + \sum_{i=1}^k \prod_{j\neq i} n_j .$$

Thus, $|\Gamma| = 1$ if and only if $\sum_{i=1}^k \prod_{j\neq i} n_j = (k - w_0) \prod_{i=1}^k n_i + 1$.

Remarks: The best-studied example is the rational double point E_8, corresponding to the solution (2, 3, 5), whose local fundamental group is the perfect extension of degree 2 of the alternating group on 5 letters. In general, in connection with the central weight w_0 it should be noted that no solution to (1) is known for which the integer $m = (\sum_{i=1}^k \prod_{j\neq i} n_j - 1)/\prod_{i=1}^k n_i$ is larger than 1.

To aid our understanding of these complex surfaces, we can model their real analogs as follows. Let $(n_1, \ldots, n_k)$ be a solution to the congruence (1). Denote by M_i the "Moebius band with n_i twists," and attach the M_i to a central Moebius band with 1 twist by the technique of plumbing. The surface under study is then the cone on the boundary of this object. The cone is a smooth two-dimensional real manifold with a singular point at the tip of the cone. See Figure 2, where the construction is illustrated for the solution (2, 3, 5).

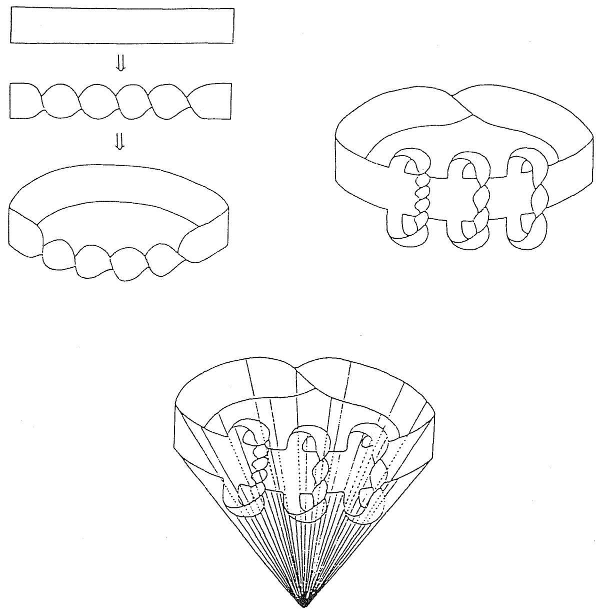

FIGURE 2

Appendix 1: The complete set of solutions to the congruence system $\prod_{j\neq i} n_j \equiv 1 \bmod (n_i)$ with 7 or fewer terms (equivalently, the complete set of solutions to the unit fraction equation $\sum_{i=1}^{k} n_i^{-1} - \prod_{i=1}^{k} n_i^{-1} = 1,\ k \leq 7$):

$k = 3$: 2,3,5

$k = 4$: 2,3,7,41
2,3,11,13

$k = 5$: 2,3,7,43,1805
2,3,7,83,85
2,3,11,17,59

$k = 6$: 2,3,7,43,1807,3263441
2,3,7,43,1811,654133
2,3,7,43,1819,252701
2,3,7,43,1825,173471
2,3,7,43,1871,51985
2,3,7,43,1901,36139
2,3,7,43,1945,25271
2,3,7,43,2053,15011
2,3,7,43,2167,10841
2,3,7,43,2501,6499
2,3,7,43,3041,4447
2,3,7,43,3611,3613
2,3,7,47,395,779729
2,3,7,47,481,2203
2,3,7,53,271,799
2,3,7,71,103,61429
2,3,11,23,31,47057

$k = 7$: 2,3,7,43,1807,3263443,10650056950805
2,3,7,43,1807,6526883,6526885
2,3,7,43,1823,193667,637617223445
2,3,7,43,1907,34165,17766223
2,3,7,43,1907,43115,163073
2,3,7,43,2159,11047,98567401
2,3,7,43,2533,7807,32435
2,3,7,43,3307,3979,642279641
2,3,7,47,395,779731,607979652629
2,3,7,47,395,779819,6832003021
2,3,7,47,395,788491,701757789
2,3,7,47,395,1559459,1559461
2,3,7,47,401,25535,1837531099
2,3,7,47,403,19403,15435513365
2,3,7,47,415,8111,6646612309
2,3,7,47,449,3299,379591
2,3,7,47,583,1223,140479765
2,3,7,55,179,24323,10057317269
2,3,7,59,163,1381,775807
2,3,7,71,103,61441,319853515
2,3,7,71,103,61477,79005919
2,3,7,71,103,61559,29133437
2,3,7,71,103,61955,7238201
2,3,7,71,103,62857,2704339
2,3,7,71,103,67213,713863
2,3,11,23,31,47059,2214502421
2,3,11,23,31,94115,94117

Appendix 2: Prime factorization of $\prod_{i=1}^{k} n_i \pm 1$ for all solutions $n_1, \ldots, n_k$ of the system of congruences (2) $\prod_{j\neq i} n_j \equiv -1 \bmod n_i$ for $k = 6$ and 7. These lists provide 380 solutions to (1) $\prod_{j\neq i} n_j \equiv 1 \bmod n_i$ with 8 terms and 1368 solutions with 9 terms, by applying Corollary 2(c). Together with solutions obtained by applying Corollary 2(b) to known solutions of 92), this gives a total of 398 solutions to (1) for $k = 8$ and 1411 solutions for $k = 9$.

$\underline{k = 6}$

$(n_i, \ldots, n_k)$	$\prod_{i=1}^{k} n_i - 1$	$\prod_{i=1}^{k} n_i + 1$
2,3,7,43,1807,3263443	$5\cdot 41\cdot 89\cdot 5119\cdot 114031$	$547\cdot 607\cdot 1033\cdot 31051$
2,3,7,43,1823,193667	$5\cdot 36931\cdot 3453019$	$37\cdot 449\cdot 38380619$
2,3,7,47,395,779731	$31\cdot 71\cdot 5939\cdot 46511$	$13\cdot 46767665587$
2,3,7,47,403,19403	$5\cdot 101\cdot 30565373$	15435513367 (prime)
2,3,7,47,415,8111	$251\cdot 269\cdot 98411$	6646612311 (prime)
2,3,7,47,583,1223	$5\cdot 29\cdot 241\cdot 40277$	1407479767 (prime)
2,3,7,55,179,24323	$9181\cdot 1095449$	$67\cdot 103\cdot 1457371$
2,3,11,23,31,47059	$19\cdot 116552759$	$19\cdot 116552759$

$\underline{k = 7}$

$(n_1, \ldots, n_k)$	$\prod_{i=1}^{k} n_i - 1$	$\prod_{i=1}^{k} n_i + 1$
2,3,7,43,1807,3263443,10650056950807	$15541\cdot 38780342479\cdot 188197244219$	$29881\cdot 67003\cdot 9119521\cdot 6212157481$
2,3,7,43,1807,3263447,213001400915	$17\cdot 240131\cdot 5556966386354188067$	$362464859\cdot 625848207273177 29$
2,3,7,43,1807,3263591,71480133827	$7477\cdot 2907138253\cdot 35023852553$	$5\cdot 1890875263\cdot 80523769616513$
2,3,7,43,1807,3264187,14298637519	$5\cdot 519\cdot 19\cdot 19267\cdot 875960006253011$	$596059\cdot 2555384970 28486753$
2,3,7,43,1823,193667,637617223447	$5849\cdot 26926271\cdot 2581441251359$	$10243\cdot 32491\cdot 1221602263409851$
2,3,7,43,3262,4051,2558951	$37\cdot 59\cdot 27983710363519$	61088439723561979 (prime)
2,3,7,43,3559,3667,33816127	$5\cdot 17\cdot 353\cdot 26563596744757$	$577\cdot 36857\cdot 37478716883$
2,3,7,47,395,779731,607979652631	369639258012703445 69529 (prime)	$14479\cdot 117594511\cdot 217096324699$
2,3,7,47,395,779831,6020372531	$191\cdot 4241\cdot 7621\cdot 592999740779$	$1\cdot 3327939478734485 06321$
2,3,7,47,403,19403,15435513367	$239\cdot 419\cdot 2379196062425981$	$1021\cdot 23335462574671 9063$
2,3,7,47,415,8111,6646612311	$31\cdot 31\cdot 71\cdot 829\cdot 15629\cdot 49942679$	$19\cdot 409\cdot 5557\cdot 1022402698813$
2,3,7,47,583,1223,1407479767	$1831\cdot 11161\cdot 96937735031$	$127\cdot 38977\cdot 400195490437$
2,3,7,55,179,24323,10057317271	$29\cdot 2311\cdot 5881\cdot 256634582371$	1011496306794975 70171 (prime)
2,3,7,67,187,283,334651	$733\cdot 67989255821$	$5\cdot 139\cdot 419\cdot 479\cdot 357281$
2,3,11,17,101,149,3109	$61819\cdot 849179$	$13\cdot 4038107431$
2,3,11,23,31,47059,2214502423	$5\cdot 4789\cdot 1970279\cdot 103946471$	$6961\cdot 1513457\cdot 4590859291$
2,3,11,23,31,47063,442938131	$37\cdot 127\cdot 208761638439227$	$5\cdot 5\cdot 7\cdot 5605548223005301$
2,3,11,23,31,47095,59897203	$19\cdot 928771\cdot 7522333121$	$7\cdot 7\cdot 109\cdot 566857\cdot 43844863$
2,3,11,23,31,47131,30382063	$43\cdot 1193\cdot 2311\cdot 8429\cdot 67433$	$5\cdot 5\cdot 3083\cdot 874266518009$
2,3,11,23,31,47243,12017087	$46062647\cdot 579990991$	$17321\cdot 23293\cdot 66217343$
2,3,11,23,31,47423,6114059	$5\cdot 59\cdot 178681\cdot 258852119$	13644326865136507 (prime)
2,3,11,23,31,49759,866923	$5\cdot 405990274405861$	$331\cdot 6132783601297$
2,3,11,23,31,60563,211031	$2017\cdot 298181849369$	$5\cdot 5\cdot 7\cdot 109\cdot 31529897257$
2,3,11,25,29,1097,2753	$7\cdot 9601\cdot 2150207$	144508961851 (prime)
2,3,11,31,35,67,369067	$17\cdot 23833\cdot 4370449$	$1553\cdot 1140203147$
2,3,13,25,29,67,2981	$2113\cdot 5345273$	$4783\cdot 2361397$

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