SOME BINOMIAL FIBONACCI IDENTITIES

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1. INTRODUCTION

Fibonacci identities are equalities of expressions made up from elements of the Fibonacci (F_n) and/or Lucas (L_n) sequences which are valid for all values of the subscript n. Fibonacci identities involving binomial coefficients are commonly referred to as *binomial Fibonacci identities* (e.g., see [1]). Perhaps the simplest and most widely used among them is

$$\sum_{h=0}^{n}\binom{n}{h}F_h = F_{2n}. \tag{1.1}$$

The aim of this paper is to use certain combinatorial identities to derive some unusual binomial Fibonacci identities.

In Section 2 we shall apply the so-called "XY-transform" [1] to the well-known (e.g., see [2]) Waring's formula

$$X^n + Y^n = \sum_{h=0}^{\lfloor n/2 \rfloor}(-1)^h D_n(h)(XY)^h(X+Y)^{n-2h} \quad (n \ge 1) \tag{1.2}$$

where

$$D_n(h) = \frac{n}{n-h}\binom{n-h}{h} \quad \left(=\binom{n-h}{h}+\binom{n-h-1}{h-1}, \text{ see [9], p. 64}\right) \tag{1.3}$$

and the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function.

In Section 3 the same technique will be applied to the combinatorial identities

$$(X+Y)^n = \sum_{h=1}^{n}\binom{2n-1-h}{n-1}(X^h+Y^h)\left(\frac{XY}{X+Y}\right)^{n-h} \quad (n \ge 1) \tag{1.4}$$

and

$$(X+Y)^n = \sum_{h=0}^{n-1} C_n(h)[X^{n-h}+(-1)^h Y^{n-h}]\left(\frac{XY}{X-Y}\right)^h \quad (n \ge 1), \tag{1.5}$$

where

$$C_n(h) = \sum_{j=0}^{h}(-1)^j\binom{h-1}{j}\binom{n}{h-j}, \tag{1.6}$$

which have been proved by L. Toscano in [10]. It has to be pointed out that (1.4)-(1.6) are quoted in [10] as due to J. G. Van de Corput.

Finally, in Section 4 we shall use the properties of the combinatorial sum

$$S_n(k,x) \stackrel{\text{def}}{=} \sum_{h=0}^{\infty}\binom{n}{kh}x^h, \tag{1.7}$$

where n and k are arbitrary positive integers, and x is an arbitrary real quantity, to obtain certain binomial Fibonacci identities, some of which involve trigonometrical expressions. Observe that the upper range indicator of the sum (1.7) has been put equal to infinity only for the sake of convenience. In fact, if n is finite, then this sum is finite as well because the binomial coefficient vanishes when $h > n/k$. Several properties of $S_n(k, x)$ have been presented by the author at the XIII Österreichischer Mathematikerkongress [3]. The detailed proofs of these properties are available in [4].

Throughout the paper $\alpha = 1-\beta = (1+\sqrt{5})/2$ denotes the golden section. The Binet forms $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$ are used without specific reference.

2. USE OF WARING'S FORMULA

Following Dresel [1], first let us put $X = \alpha^k$ and $Y = \beta^k$ [whence $XY = (-1)^k$ and $X+Y = L_k$] in (1.2), thus getting the expression of L_{nk} as a polynomial in L_k:

$$L_{nk} = \sum_{h=0}^{\lfloor n/2 \rfloor} (-1)^{(k+1)h} D_n(h) L_k^{n-2h} \quad (n \geq 1). \tag{2.1}$$

Then let us put $X = \alpha^k$ and $y = -\beta^k$ [whence $XY = (-1)^{k+1}$ and $X+Y = \sqrt{5}F_k$] in (1.2), thus obtaining

$$L_{nk} = \sum_{h=0}^{n/2} (-1)^{hk} D_n(h) 5^{n/2-h} F_k^{n-2h} \quad (n \geq 2 \text{ even}), \tag{2.2}$$

and

$$F_{nk} = \sum_{h=0}^{(n-1)/2} (-1)^{hk} D_n(h) 5^{(n-1)/2-h} F_k^{n-2h} \quad (n \geq 1 \text{ odd}). \tag{2.3}$$

Observe that the identity (2.3), the proof of which is nothing but a trivial replacement, is an equivalent form of the statement of Theorem 1 of [8].

3. USE OF TOSCANO'S FORMULAS

We shall confine ourselves to give two simple examples of application of formulas (1.4) and (1.5).

Letting $X = \alpha^k$ and $Y = \beta^k$ in (1.4) yields

$$L_k^n = \sum_{h=1}^{n} (-1)^{k(n-h)} \binom{2n-1-h}{n-1} \frac{L_{hk}}{L_k^{n-k}} \quad (n \geq 1) \tag{3.1}$$

whence, after multiplying both sides by L_k^n, we obtain

$$L_k^{2n} = (-1)^{kn} \sum_{h=1}^{n} \binom{2n-1-h}{n-1} L_{hk} L_k^h \quad (n \geq 1). \tag{3.2}$$

Observe that, for $k = 1$, the identity (3.2) reduces to

$$\sum_{h=1}^{n} (-1)^{n+h} \binom{2n-1-h}{n-1} L_h = 1 \quad (n \geq 1). \tag{3.3}$$

Letting $X = \alpha^k$ and $y = -\beta^k$ in (1.5) yields

$$(\sqrt{5}F_k)^n = \sum_{h=0}^{n-1} C_n(h)\left[\alpha^{k(n-h)} + (-1)^h(-1)^{k(n-h)}\right]\frac{(-1)^{(k+1)h}}{L_k^h} \quad (n \geq 1) \tag{3.4}$$

whence

$$F_k^n = \frac{1}{5^{\lfloor n/2 \rfloor}} \sum_{h=0}^{n-1} (-1)^{(k+1)h} C_n(h) \frac{A_{k(n-h)}}{L_k^h} \quad (n \geq 1), \tag{3.5}$$

where A stands for L (F) when n is even (odd). Observe that, for $k = 1$, the identity (3.5) reduces to

$$\sum_{h=0}^{n-1} C_n(h) A_{n-h} = 5^{\lfloor n/2 \rfloor} \quad (n \geq 1). \tag{3.6}$$

4. USE OF $S_n(k, x)$

First, we recall some of the results established in [3] and [4], then we use them to obtain some binomial Fibonacci identities. The following notation is used throughout this section:

(i) $i = \sqrt{-1}$: the imaginary unit,

(ii) $r_s(k) = e^{i2\pi(s-1)/k}$: the s^{th} $(s = 1, 2, \ldots, k)$ of the k distinct k^{th} roots of 1,

(iii) $x^{1/k}$ (or $\sqrt{x}$ in the case $k = 2$): the principal value of the k^{th} root of x,

(iv) $\operatorname{atan} x$: the principal branch of the function $\tan^{-1} x$ $(-\pi/2 < \operatorname{atan} x < \pi/2)$.

We point out that $\operatorname{atan} x$ is the value of $\tan^{-1} x$ one obtains by means of the common pocket calculators. The results obtained in this section can then be readily checked.

4.1 Some Results Concerning $S_n(k, x)$ and Certain Related Sums

The following identity is perhaps the main result established in [3] and [4]:

$$S_n(k, x) = \frac{1}{k}\sum_{s=1}^{k}\left[1 + r_s(k)x^{1/k}\right]^n. \tag{4.1}$$

For the convenience of the reader, we report the short and elegant proof of (4.1) that has been suggested by the referee.

Proof of (4.1):

$$\begin{aligned}\sum_{s=1}^{k}\left[1 + x^{1/k}e^{i2\pi(s-1)/k}\right]^n &= \sum_{s=1}^{k}\sum_{h=0}^{n}\binom{n}{h}\left[xe^{i2\pi(s-1)}\right]^{h/k} \\ &= \sum_{h=0}^{n}\binom{n}{h}x^{h/k}\sum_{s=1}^{k}\left[e^{i2\pi h/k}\right]^{s-1} \qquad (4.2) \\ &= k\sum_{h=0}^{\infty}\binom{n}{kh}x^h,\end{aligned}$$

since the summation (4.2) is just the sum of a geometric progression equal to k if $k \mid h$, and zero otherwise. Q.E.D.

Observe that, for k = 2 and 3, the identity (4.1) is reported on pages 123 and 135 of [9], respectively. The extension of (4.1) to negative values of n (e.g., see [9], p. 1) yields

$$S_{-n}(k,x)=\sum_{h=0}^{\infty}\binom{-n}{kh}x^h=\sum_{h=0}^{\infty}(-1)^{kh}\binom{n-1+kh}{kh}x^h$$
$$=\frac{1}{k}\sum_{s=1}^{k}\left[1+r_s(k)x^{1/k}\right]^{-n}\quad(\text{if }|x|<1). \tag{4.3}$$

Moreover, we consider the combinatorial sums

$$S_n^{(1)}(k,x)\overset{\text{def}}{=}\sum_{h=0}^{\infty}h\binom{n}{kh}x^h=\frac{nx^{1/k}}{k^2}\sum_{s=1}^{k}r_s(k)\left[1+r_s(k)x^{1/k}\right]^{n-1}, \tag{4.4}$$

$$R_n(x)\overset{\text{def}}{=}\sum_{h=0}^{\infty}\binom{n}{2h+1}x^h=\left[(1+\sqrt{x})^n-(1-\sqrt{x})^n\right]/(2\sqrt{x})\quad(x\neq 0). \tag{4.5}$$

Special cases of (4.1) and (4.3)-(4.5) which interest us are:

$$S_n(2,2)=Q_n/2\quad(Q_n,\text{ the }n^{\text{th}}\text{ Pell-Lucas number [7]}); \tag{4.6}$$

$$S_n(2,5)=2^{n-1}L_n; \tag{4.7}$$

$$S_n^{(1)}(2,5)=5n2^{n-3}F_{n-1}; \tag{4.8}$$

$$R_n(5)=2^{n-1}F_n. \tag{4.9}$$

Using the identity (4.1) along with the polar form for complex numbers yields the following trigonometrical expressions for $S_n(k,x)$ ($k=2,3,$ and 4).

$$S_n(2,x)=(1-x)^{n/2}\cos(n\operatorname{atan}\sqrt{-x})\quad(x<0), \tag{4.10}$$

$$S_n(3,x)=\frac{1}{3}\left[(1+x^{1/3})^n+2X^{n/2}\cos\left(n\operatorname{atan}\frac{\sqrt{3}x^{1/3}}{2-x^{1/3}}\right)\right]\quad(x<8)$$
$$(\text{where }X=x^{2/3}-x^{1/3}+1), \tag{4.11}$$

$$S_n(4,x)=\frac{1}{4}\left[(1+x^{1/4})^n+(1-x^{1/4})^n+2(1+\sqrt{x})^{n/2}\cos(n\operatorname{atan}x^{1/4})\right]\quad(x\geq 0). \tag{4.12}$$

Some special cases of (4.10)-(4.12) are:

$$S_n(2,-1)=2^{n/2}\cos(n\pi/4); \tag{4.13}$$

$$S_n(3,1)=[2^n+2\cos(n\pi/3)]/3\quad(\text{cf. }0.152\text{-}1\text{ of [5]}); \tag{4.14}$$

$$S_n(3,-1)=[3^{n/2}2\cos(n\pi/6)]/3; \tag{4.15}$$

$$S_n(4,1)=[2^{n-1}+2^{n/2}\cos(n\pi/4)]/2\quad(\text{cf. }0.153\text{-}1\text{ of [5]}); \tag{4.16}$$

$$S_n(4,25)=2^{n-2}L_n+\frac{6^{n/2}}{2}\cos(n\operatorname{atan}\sqrt{5}). \tag{4.17}$$

4.2 Some Fibonacci Identities

By using (4.1), (4.3), and (4.4) along with the Binet forms for Fibonacci and Lucas numbers, and some usual identities available in [6], pages 52-60, a great variety of somewhat unusual Fibonacci identities can be obtained, a small sample of which is reported in the sequel. To save space, only one of them will be proved in full detail. Particular emphasis is given to the use of $S_n(2, x)$. It can be noted that the special cases (4.7)-(4.9) and (4.17) are themselves Fibonacci identities.

4.2.1 Results

$$\sum_{h=0}^{\infty} h\binom{n}{h}F_h = nF_{2n-1}. \tag{4.18}$$

$$\sum_{h=0}^{\infty}(-1)^h\binom{n-1+h}{h}\frac{F_h}{2^h} = -2^n B_n / 5^{\lfloor (n+1)/2 \rfloor}, \tag{4.19}$$

where B stands for F (L) when n is even (odd). Observe that letting $n = 1$ and 2 in (4.19), and combining the results yield the remarkable equality

$$\sum_{h=0}^{\infty}(-1)^h F_h / 2^h = \sum_{h=0}^{\infty}(-1)^h hF_h / 2^h \ \ (= -2/5), \tag{4.20}$$

which shows how the sum on the left-hand side is unconcerned at the introduction of the factor h.

$$\sum_{h=0}^{\infty}\binom{n}{2h}F_{2h} = (F_{2n} - F_n)/2. \tag{4.21}$$

$$\sum_{h=0}^{\infty} h\binom{n}{2h}F_{2h} = n(F_{2n-1} - F_{n-2})/4. \tag{4.22}$$

$$\sum_{h=0}^{\infty} h\binom{n}{2h}F_{2h}^2 = \begin{cases} n[5^{n/2}F_{n+1} + L_{n+1} - 2^n]/20 & (n \text{ even}), \\ n[(5^{(n-1)/2} - 1)\,L_{n+1} - 2^n]/20 & (n \text{ odd}). \end{cases} \tag{4.23}$$

$$\sum_{h=0}^{\infty}\binom{n}{2h}F_h^2 = \frac{1}{10}\left[L_{2n} + L_n - 2^{(n+4)/2}\cos\frac{n\pi}{4}\right]. \tag{4.24}$$

$$\sum_{h=0}^{\infty} h\binom{n}{2h}F_h^2 = \frac{n}{20}\left[L_{2n-1} + L_{n-2} + 2^{(n+3)/2}\sin\frac{(n-1)\pi}{4}\right]. \tag{4.25}$$

$$\sum_{h=0}^{\infty}\binom{n}{3h}F_{3h} = \frac{F_{2n}}{3} + \frac{2^{(n+4)/2}}{3\sqrt{5}}\sin\left(\frac{n\pi}{6}\right)\sin\left[\frac{n(\operatorname{atan}\sqrt{15} - \pi)}{2}\right]. \tag{4.26}$$

Observe that the right-hand side of (4.26) reduces to $F_{2n}/3$ whenever $n \equiv 0 \pmod 6$.

$$\sum_{h=0}^{\infty}\binom{n}{4h}F_{4h} = \frac{F_{2n} - F_n}{4} + \frac{5^{n/4}}{2}F_{n/2}\cos(n \operatorname{atan}\alpha) \ \ (n \text{ even}). \tag{4.27}$$

4.2.2 A Proof

Proof of (4.27): By (4.12) and taking into account that the principal value of $(\beta^4)^{1/4}$ is $-\beta = (\sqrt{5}-1)/2$, let us rewrite the left-hand side of (4.27) as

$$\begin{aligned}
X_n &= \frac{1}{\sqrt{5}}[S_n(4, \alpha^4) - S_n(4, \beta^4)] \\
&= \frac{1}{\sqrt{5}}\left(\frac{1}{4}[(1+\alpha)^n + (1-\alpha)^n + 2(1+\alpha^2)^{n/2}\cos(n \operatorname{atan}\alpha)]\right. \\
&\quad \left. -\frac{1}{4}\{(1-\beta)^n + (1+\beta)^n + 2(1+\beta^2)^{n/2}\cos[n \operatorname{atan}(-\beta)]\}\right) \\
&= \frac{F_{2n}-F_n}{4} + \frac{1}{2\sqrt{5}}\{(1+\alpha^2)^{n/2}\cos(n \operatorname{atan}\alpha) - (1+\beta^2)^{n/2}\cos[n \operatorname{atan}(-\beta)]\} \\
&= \frac{F_{2n}-F_n}{4} + \frac{1}{2\sqrt{5}}\{(\sqrt{5}\alpha)^{n/2}\cos(n \operatorname{atan}\alpha) - (-\sqrt{5}\beta)^{n/2}\cos[n \operatorname{atan}(-\beta)]\} \\
&= \frac{F_{2n}-F_n}{4} + \frac{5^{(n-2)/4}}{2}\{\alpha^{n/2}\cos(n \operatorname{atan}\alpha) - (-\beta)^{n/2}\cos[n \operatorname{atan}(-\beta)]\}.
\end{aligned} \tag{4.28}$$

The identity (4.28) is valid for all positive n. For n even, (4.28) simplifies remarkably. Consider the trigonometrical identity

$$\operatorname{atan} x + \operatorname{atan}(1/x) = \pi/2, \tag{4.29}$$

whence

$$\operatorname{atan}(-\beta) = \pi/2 - \operatorname{atan}\alpha, \tag{4.30}$$

and replace (4.30) in (4.28), thus getting the identity

$$X_n = \frac{F_{2n}-F_n}{4} + \frac{5^{(n-2)/4}}{2}\left[\alpha^{n/2}\cos(n \operatorname{atan}\alpha) - (-\beta)^{n/2}\cos\left(\frac{n\pi}{2} - n \operatorname{atan}\alpha\right)\right]. \tag{4.31}$$

Recalling that

$$\cos\left(\frac{n\pi}{2} - x\right) = \begin{cases} \cos x, & \text{if } n \equiv 0 \pmod 4, \\ -\cos x, & \text{if } n \equiv 2 \pmod 4, \end{cases} \tag{4.32}$$

the identity (4.31) becomes

$$X_n = \frac{F_{2n}-F_n}{4} + \frac{5^{(n-2)/4}}{2}(\alpha^{n/2} - \beta^{n/2})\cos(n \operatorname{atan}\alpha) \quad (n \text{ even}), \tag{4.33}$$

whence (4.27) is immediately obtained. Q.E.D.

5. CONCLUDING REMARKS

Several binomial Fibonacci identities, most of which we believe to be new, have been established in this paper. In our eyes, the most interesting among them are those derived from the specialization of the combinatorial sum $S_n(k, x)$ to the cases $1 \le k \le 4$.

These identities represent only a small sample of the possibilities available to us. In fact, apart from the extension of the study to values of k greater than 4, the results obtained in Section 4 can apply *mutatis mutandis* to second-order recurring sequences other than the Fibonacci sequence. For instance, the analog of (4.27) for Pell numbers P_n is

$$\sum_{h=0}^{\infty}\binom{n}{4h}P_{4h} = 2^{(n-4)/2}P_n + (-1)^{n/2}2^{(3n-4)/4}P_{n/2}\cos\frac{n\pi}{8} \quad (n \text{ even}), \tag{5.1}$$

which reduces to $2^{(n-4)/2}P_n$ when $n \equiv 4 \pmod 8$.

Moreover, the sequences $S_n(k, m, x)$ defined as

$$S_n(k, m, x) \stackrel{\text{def}}{=} \sum_{h=0}^{\infty}\binom{n}{kh}S_h(m, x) \tag{5.2}$$

seem to be worthy of thorough investigation. This will be the goal of a future work. As a minor illustration, we leave the interested reader the proof of the identity

$$S_n(2, 2, 1) = (2 + Q_n)/4 \tag{5.3}$$

which involves the Pell-Lucas numbers.

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