REMAINDER FORMULAS INVOLVING GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

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1. INTRODUCTION

Synthetic division schemes to calculate the linear remainder when a polynomial is divided by a quadratic are used in numerical algorithms, such as Bairstow's method, for finding quadratic factors of polynomials. In this paper formulas for the linear remainder are derived in terms of the coefficients of the polynomial and coefficients of the quadratic divisor. These take a particularly compact form when expressed in terms of generalized Fibonacci and Lucas polynomials. Three different forms of the remainder are considered and second-order linear partial differential equations are introduced which have the linear remainder coefficients as solutions. Ordinary differential equations satisfied by two families of Fibonacci and Lucas polynomials are derived using identities which relate them to the generalized polynomials, and nonpolynomial solutions are deduced from corresponding solutions of the partial differential equations.

For any polynomial it is proved there exists a two-variable potential function with the property that its critical points correspond to coefficients in the quadratic factors of the polynomial. The potential function is defined by the linear remainder coefficients and an explicit formula is obtained in terms of the coefficients of the polynomial and generalized Lucas polynomials.

Hoggatt and Long [6] and Frei [2] have shown the generalized Fibonacci polynomials $F_n(x, y)$ and generalized Lucas polynomials $L_n(x, y)$ satisfy the Binet formulas

$$F_n(x, y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n(x, y) = \alpha^n + \beta^n$$

for $n \geq 0$ where α and β are the zeros of $t^2 - xt - y$ so that

$$\alpha = \tfrac{1}{2}(x + \sqrt{x^2 + 4y}), \quad \beta = \tfrac{1}{2}(x - \sqrt{x^2 + 4y})$$

and

$$\alpha + \beta = x, \quad \alpha\beta = -y, \quad \alpha - \beta = \sqrt{x^2 + 4y}.$$

These formulas are used to obtain many of the results in the following sections.

2. REMAINDER FORMULAS

The remainder coefficients $F(x, y)$ and $G(x, y)$ are defined by

$$P(t) = (t^2 - xt - y)Q(t) + F(x, y)(t - x) + G(x, y), \tag{1}$$

where $Q(t)$ is the quotient when a polynomial $P(t)$ is divided by $t^2 - xt - y$. Let

$$\begin{aligned} P(t) &= a_N t^N + a_{N-1} t^{N-1} + \cdots + a_1 t + a_0, \\ Q(t) &= b_N t^{N-2} + b_{N-1} t^{N-3} + \cdots + b_3 t + b_2, \end{aligned} \quad (N \geq 2)$$

and equate coefficients of powers of t in (1), giving the recurrence relation

$$\begin{aligned} &b_N = a_N, \quad b_{N-1} = a_{N-1} + xb_N, \\ &b_n = a_n + xb_{n+1} + yb_{n+2}, \end{aligned} \qquad (n = N-2, N-3, \ldots, 1, 0),$$

where $F(x, y) = b_1$ and $G(x, y) = b_0$. These form the basis for the synthetic division calculation of the remainder coefficients, described by Mathews [7], for numerical values of x and y.

Although these recurrence relations can be used to generate expressions for b_1 and b_0, it is simpler to obtain explicit formulas for the remainder coefficients by substituting $t = \alpha$ and $t = \beta$ in (1) giving

$$P(\alpha) = -\beta F(x, y) + G(x, y), \quad P(\beta) = -\alpha F(x, y) + G(x, y), \tag{2}$$

respectively. It follows that

$$F(x, y) = \frac{P(\alpha) - P(\beta)}{\alpha - \beta}, \quad G(x, y) = \frac{\alpha P(\alpha) - \beta P(\beta)}{\alpha - \beta},$$

and, using the Binet formulas, these can be further simplified to

$$F(x, y) = \sum_{n=0}^{N} a_n F_n(x, y), \quad G(x, y) = \sum_{n=0}^{N} a_n F_{n+1}(x, y).$$

If the linear remainder in (1) is taken instead as $F(x, y)t + H(x, y)$, which is the form used by Fröberg [3], then $H(x, y) = G(x, y) - xF(x, y)$ and it can be shown that

$$H(x, y) = \frac{\alpha P(\beta) - \beta P(\alpha)}{\alpha - \beta} = a_0 + y\sum_{n=1}^{N} a_n F_{n-1}(x, y).$$

Similarly, by taking the remainder as $F(x, y)(t - \frac{1}{2}x) + L(x, y)$, we have $L(x, y) = G(x, y) - \frac{1}{2}xF(x, y)$ and $L(x, y) = \frac{1}{2}[P(\alpha) + P(\beta)] = \frac{1}{2}\sum_{n=0}^{N} a_n L_n(x, y)$.

3. DIFFERENTIAL EQUATIONS

Consider the linear partial differential equations

$$\frac{\partial^2 \ell}{\partial x^2} - x\frac{\partial^2 \ell}{\partial x \partial y} - y\frac{\partial^2 \ell}{\partial y^2} = \frac{\partial \ell}{\partial y} \tag{3}$$

$$\frac{\partial^2 f}{\partial x^2} - x\frac{\partial^2 f}{\partial x \partial y} - y\frac{\partial^2 f}{\partial y^2} = 2\frac{\partial f}{\partial y}, \tag{4}$$

$$\frac{\partial^2 h}{\partial x^2} - x\frac{\partial^2 h}{\partial x \partial y} - y\frac{\partial^2 h}{\partial y^2} = -\frac{x}{y}\frac{\partial h}{\partial x}. \tag{5}$$

The change of variables from x, y, to α, β in the region $x^2 + 4y > 0$, where (3)-(5) are classified as hyperbolic, transforms them into their canonical form.

It can be shown, using the chain rule, that the canonical form of (3) is

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = 0,$$

from which the general solution is $\ell = \ell_1(\alpha) + \ell_2(\beta)$. Substituting $\ell_1(\alpha) = \alpha^n$ and $\ell_2(\beta) = \beta^n$ gives $\ell = L_n(x, y)$ as a solution of (3), and the principle of superposition means the remainder coefficient $L(x, y) = \sum_{n=0}^{N} a_n L_n(x, y)$ is also a solution. Substituting $\ell_1(\alpha) = \alpha^n$ and $\ell_2(\beta) = -\beta^n$ gives $\ell = \sqrt{x^2+4y}F_n(x, y)$; hence, $\sqrt{x^2+4y}\sum_{n=0}^{N} c_n F_n(x, y)$ is another solution of (3) for arbitrary constants c_n.

Similarly, the canonical form of (4) can be derived as

$$\frac{\partial^2}{\partial\alpha\partial\beta}[(\alpha-\beta)f] = 0,$$

from which it can be deduced that the remainder coefficients

$$F(x, y) = \sum_{n=0}^{N} a_n F_n(x, y), \quad G(x, y) = \sum_{n=0}^{N} a_n F_{n+1}(x, y), \text{ and also } \sum_{n=0}^{N} c_n L_n(x, y) / \sqrt{x^2+4y}$$

are solutions of (4).

The canonical form of (5) is

$$\frac{\partial^2}{\partial\alpha\partial\beta}\left[\frac{\alpha-\beta}{\alpha\beta}h\right] = 0,$$

which can be shown to have as solutions the remainder coefficient

$$H(x, y) = a_0 + y\sum_{n=1}^{N} a_n F_{n-1}(x, y) \quad \text{and} \quad \left[c_0 x + y\sum_{n=1}^{N} c_n L_{n-1}(x, y)\right]\Big/\sqrt{x^2+4y}.$$

The above solutions also apply in the region $x^2+4y<0$, where (3)-(5) are classified as elliptic, although it would be appropriate to replace $\sqrt{x^2+4y}$ by $\sqrt{-x^2-4y}$ in the nonpolynomial solutions.

It is not the purpose of this paper to investigate all solutions of (3)-(5), but it is not difficult to see that definition of $F_n(x, y)$ and $L_n(x, y)$ for $n<0$ (see [2]) and allowing infinite sums, subject to any convergence conditions being satisfied, would produce further solutions.

The single variable polynomials $F_n(1, z)$ and $L_n(1, z)$, $n \geq 0$, with the properties

$$F_n(x, y) = x^{n-1}F_n(1, z), \tag{6}$$

$$L_n(x, y) = x^n L_n(1, z), \tag{7}$$

where $z = y/x^2$, are referred to as the Fibonacci and Lucas polynomials, respectively, by Doman and Williams [1]. Galvez and Devesa [4] have shown that they satisfy the ordinary differential equations

$$z(1+4z)\frac{d^2F_n}{dz^2} - [(n-1)+2(2n-5)z]\frac{dF_n}{dz} + (n-1)(n-2)F_n = 0, \tag{8}$$

$$z(1+4z)\frac{d^2L_n}{dz^2} - [(n-1)+2(2n-3)z]\frac{dL_n}{dz} + n(n-1)L_n = 0, \tag{9}$$

which may also be proved by substituting (6) and (7) into (4) and (3), respectively.

Using the earlier results, it can be shown that a second linearly independent solution of (8) is $L_n(1, z) / \sqrt{|1+4z|}$, and a second linearly independent solution of (9) is $\sqrt{|1+4z|}\, F_n(1, z)$.

The polynomials $F_n(u, 1)$ and $L_n(u, 1)$, also referred to as Fibonacci and Lucas polynomials by Hoggatt and Bicknell [5], are related to the generalized polynomials by

$$F_n(x, y) = y^{(n-1)/2} F_n(u, 1), \quad L_n(x, y) = y^{n/2} L_n(u, 1),$$

where $u = x / \sqrt{y}$. Substitution into (4) and (3) shows they satisfy

$$(4+u^2)\frac{d^2 F_n}{du^2} + 3u\frac{dF_n}{du} - (n^2-1)F_n = 0,$$

$$(4+u^2)\frac{d^2 L_n}{du^2} + u\frac{dL_n}{du} - n^2 L_n = 0,$$

which also have solutions $L_n(u, 1) / \sqrt{u^2+4}$ and $\sqrt{u^2+4}\, F_n(u, 1)$, respectively.

4. A POTENTIAL FUNCTION

Differentiating (1) with respect to x and rearranging gives

$$tQ(t) = (t^2 - xt - y)\frac{\partial Q(t)}{\partial x} + \frac{\partial F}{\partial x}(t-x) + \frac{\partial G}{\partial x} - F, \tag{10}$$

whereas, differentiating (1) with respect to y, multiplying by t and rearranging gives

$$tQ(t) = (t^2 - xt - y)\left[t\frac{\partial Q(t)}{\partial y} + \frac{\partial F}{\partial y}\right] + \frac{\partial G}{\partial y}(t-x) + x\frac{\partial G}{\partial y} + y\frac{\partial F}{\partial y}. \tag{11}$$

Comparing (10) and (11) gives

$$\frac{\partial F}{\partial x} = \frac{\partial G}{\partial y}, \tag{12}$$

$$\frac{\partial G}{\partial x} = F + x\frac{\partial G}{\partial y} + y\frac{\partial F}{\partial y}. \tag{13}$$

Equation (12) is the condition for the existence of a $\phi(x, y)$, defined as the potential function of $P(t)$ and also denoted by $\phi[P(t)]$, with the properties

$$\frac{\partial \phi}{\partial x} = G(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = F(x, y). \tag{14}$$

Substituting (14) into (13) proves that ϕ satisfies (3); hence, it may be expressed in the form $\ell_1(\alpha) + \ell_2(\beta)$. It is easily shown, using the chain rule (14) and (2), that

$$\frac{\partial \phi}{\partial \alpha} = P(\alpha) \quad \text{and} \quad \frac{\partial \phi}{\partial \beta} = P(\beta),$$

and therefore,

$$\phi = \sum_{n=0}^{N} \frac{a_n}{n+1}(\alpha^{n+1} + \beta^{n+1}) = \sum_{n=0}^{N} \frac{a_n}{n+1} L_{n+1}(x, y),$$

where $\phi[0]$ is defined to be zero.

If $F(x^*, y^*) = G(x^*, y^*) = 0$, then from (1) and (14) it follows that the polynomial $P(t)$ has a quadratic factor $t^2 - x^*t - y^*$ if and only if (x^*, y^*) is a critical point of its potential function $\phi[P(t)]$. Obviously any linear combination of generalized Lucas polynomials, excluding the constant $L_0(x, y)$, may be considered as the potential function of some polynomial. In the case when this polynomial has N distinct real roots, its potential function has $N(N-1)/2$ critical points all deducible from pairwise multiplication of the linear factors of the polynomial. Then calculation of the roots, say by the Newton-Raphson method, would generally be a computationally efficient way of finding the larger number of critical points of the linear combination of generalized Lucas polynomials when $N \geq 4$.

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