A NOTE ON MULTIPLICATIVE PARTITIONS OF BIPARTITE NUMBERS

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1. INTRODUCTION

For a positive integer n, let $f(n)$ be the number of essentially different ways of writing n as a product of factors greater than 1, where two factorizations of a positive integer are said to be essentially the same if they differ only in the order of the factors. For example, $f(12)=4$, since $12=2\cdot 6=3\cdot 4=2\cdot 2\cdot 3$. This function was introduced by Hughes and Shallit [1], who proved that $f(n)\le 2n^{\sqrt{2}}$ for all n. Mattics and Dodd [2] improved the inequality so that $f(n)\le n/\log n$ for all $n>1, n\ne 144$. Landman and Greenwell [3] generalized the notion of multiplicative partitions to bipartite numbers. For positive integers m and n, $mn>1$, let $f_2(m,n)$ denote the number of essentially different ways of writing the pair (m,n) as a product $\prod_{1\le i\le k}(a_i, b_i)$, where $a_ib_i>1$ for $1\le i\le k$ and where multiplication is done coordinate-wise. Similarly, for positive integers m and n, $mn>1$, let $g(m,n)$ be the number of essentially different ways of writing the pair (m,n) as a product $\prod_{1\le i\le k}(a_i, b_i)$, where $a_i>1$, $b_i>1$ for $1\le i\le k$. Let $g(1,1)=f_2(1,1)$ be 1. For example, $f_2(6,2)=5$, since $(6,2)=(6,1)(1,2)=(3,2)(2,1)=(3,1)(2,2)=(3,1)(1,2)(2,1)$ and $g(6,4)=2$, since $(6,4)=(3,2)(2,2)$. In a recent paper [3], Landman and Greenwell proved that

$$f_2(m,n)<\frac{(mn)^{1.516}}{\log(mn)},$$

and they conjectured that 1.516 can be replaced by 1.251. In this paper we approximate $g(m,n)$ by a completely multiplicative function $h(mn)$. Using this approximation, we prove that

$$f_2(m,n)<(2160)^2(mn)^{1.143}.$$

We also prove that $f_2(m,n)<(mn)^{1.251}/\log(mn)$ for $mn\ge 10^{83}$.

2. NOTATIONS

For convenience, we will define some notations and conventions used in this paper. Let N denote the set of all positive integers and p_i denote the i^{th} prime (i.e., $p_i=2, p_2=3$, etc.). The prime factorizations of $m>1$ and $n>1$ may be considered as $m=\prod_{i=1}^t q_i^{\alpha_i}$, $n=\prod_{j=1}^s s_i^{\beta_j}$, where $\{q_i\}$ are the distinct prime factors of m, $\{s_j\}$ are the distinct prime factors of n, and $\{\alpha_i\}$, $\{\beta_j\}$ are nonincreasing sequences of positive integers. Let $\hat{m}=\prod_{i=1}^t p_i^{\alpha_i}\le m$ and $\hat{n}=\prod_{j=1}^s p_j^{\beta_j}\le n$. Then, clearly, $f_2(m,n)=f_2(\hat{m},\hat{n})$. Hence, let $M=\{a\in N \mid a=\prod_{i=1}^k p_i^{\theta_i}>1$, where $\{\theta_i\}$ is a nonincreasing sequence of positive integers and $k\in N\}$. The completely multiplicative functions h and T are defined on N as follows:

(a) $T(1) = 1$; $T(2) = (7/4)$; $T(3) = (11/4)$; $T(p_r) = (r + 7/4)$ for $r \geq 3$; $T(ab) = T(a)T(b)$ for a, $b \in N$

(b) $h(1) = 1$; $h(p_i) = r_i$, where $\{r_i\}_{i\geq 1}$ is the sequence of real numbers defined by

$$r_{i+1} = 1 + \prod_{j=1}^{i} \frac{r_j}{r_j - 1} \sqrt{1 + \left(\prod_{k=1}^{i} \frac{r_k}{r_k - 1} - 1 \right)^2} \quad \text{for } i \geq 1 \text{ and } r_i = 2;$$

$h(ab) = h(a)h(b)$ for $a, b \in N$.

For any positive integer k, the multiplicative function $d^{(k)}$ is defined on N as follows:

$$d^{(k)}(a) = \sum_{\substack{\ell \mid a \\ p_i \nmid \ell \text{ for all } i \geq k}} 1$$

[i.e., $d^{(k)}(p_i^b) = 1$ for $i \geq k$; $d^{(k)}(p_i^b) = b + 1$ for $i < k$].

3. PROOF OF THE MAIN RESULT

Throughout this paper, all variables represent nonnegative integers, unless otherwise specified. The following lemma will be used frequently in the remainder of our work.

Lemma 1: $r_i + 2 < r_{i+1} < r_i + 2.5$ if $i \geq 7$.

Proof: Fix $i \geq 6$ and let $y = \prod_{j=1}^{i} r_j / (r_j - 1)$. Then $y > 4$ and

$$\begin{aligned} r_{i+2} &= y\left(1 + \frac{1}{y\sqrt{(y-1)^2 + 1}}\right)\sqrt{\left(y - 1 + \frac{1}{\sqrt{(y-1)^2+1}}\right)^2 + 1} + 1 \\ &< \left(y + \frac{1}{\sqrt{(y-1)^2+1}}\right)\left(\sqrt{(y-1)^2+1} + \frac{1}{\sqrt{(y-1)^2+1}}\right) + 1 \\ &= r_{i+1} + \frac{y}{\sqrt{(y-1)^2+1}} + 1 + \frac{1}{(y-1)^2+1} < r_{i+1} + 2.5. \end{aligned}$$

Similarly, one can prove that $r_{i+2} > r_{i+1} + 2$. Q.E.D.

Lemma 2: If $m = \prod_{i=1}^{t} p_i^{\alpha_i} \in M$ and $1 \leq s \leq t$, then

$$\sum_{\ell \mid m} \frac{d^{(s)}(\ell)}{h(\ell)} \leq \frac{r_t}{r_t - 1} \cdot \prod_{i=1}^{t-1} \left(\frac{r_i}{r_i - 1} \right)^2.$$

Proof: From Lemma 1, we know that $r_i > 1$ for all $i \geq 1$. Then we have

$$\begin{aligned} \sum_{\ell \mid m} \frac{d^{(s)}(\ell)}{h(\ell)} &= \prod_{i=1}^{s-1} \left(\sum_{j=0}^{\alpha_i} \frac{j+1}{r_i^j} \right) \cdot \prod_{a=s}^{t} \left(\sum_{k=0}^{\alpha_a} \frac{1}{r_a^k} \right) \leq \prod_{i=1}^{s-1} \left(\sum_{j=0}^{\infty} \frac{j+1}{r_i^j} \right) \cdot \prod_{a=s}^{t} \left(\sum_{k=0}^{\infty} \frac{1}{r_a^k} \right) \\ &= \prod_{i=1}^{s-1} \left(\frac{r_i}{r_i - 1} \right)^2 \cdot \prod_{a=s}^{t} \left(\frac{r_a}{r_a - 1} \right). \quad \text{Q.E.D.} \end{aligned}$$

With the aid of Lemma 2, we establish an upper bound on $g(m,n)$.

Proposition 1: The function $g(m,n)$ satisfies the inequality:

$$g(m,n) \le h(m)\cdot h(n) = \left(\prod_{i=1}^{t} r_i^{\alpha_i}\right)\left(\prod_{j=1}^{s} r_j^{\beta_j}\right),$$

where $m=\prod_{i=1}^{t} p_i^{\alpha_i}$ and $n=\prod_{j=1}^{s} p_j^{\beta_j}$.

Proof: It is enough to show that $g(m,n)\le h(mn)$ for $m,n\in M$, since, for any positive integers $a=\prod_{i=1}^{c} q_i^{a_i}$ and $b=\prod_{j=1}^{d} s_j^{b_j}$,

$$g(a,b) = g\left(\prod_{i=1}^{c} p_i^{a_i}, \prod_{j=1}^{d} p_j^{b_j}\right) \quad \text{and} \quad h\left(\prod_{i=1}^{c} p_i^{a_i}\right)h\left(\prod_{j=1}^{d} p_j^{b_j}\right) \le h(a)h(b),$$

where $\{q_i\}$ are the distinct prime factors of a, $\{s_j\}$ are the distinct prime factors of b, and $\{a_i\}$, $\{b_i\}$ are nonincreasing sequences of positive integers. The statement clearly holds for the case $n\le 2$, since $g(m,1)=0$ for $m>1$. Hence, without loss of generality, we may assume $m\ge n>2$. Let $m'=m/p_t$ and $n'=n/p_s$. First, we introduce some sets:

$$S=\{\{(a_i,b_i)\}_{1\le i\le e} \mid (1)\ (m,n)=\textstyle\prod_{1\le i\le e}(a_i,b_i),\ (2)\ a_j,b_j\ge 2 \text{ for all } 1\le j\le e,$$
$$(3)\ a_j\ge a_{j+1}; \text{ and if } a_j=a_{j+1}, \text{ then } b_j\ge b_{j+1} \text{ for all } 1\le j\le e-1\};$$

$$A(\ell,k)=\{\{(a_i,b_i)\}_{1\le i\le e}\in S \mid (a_{i_0},b_{i_0})=(p_t\ell,p_s k) \text{ for some } 1\le i_0\le e\};$$

$$B(\ell,k)=\{\{(a_i,b_i)\}_{1\le i\le e}\in S \mid p_t\nmid a_{i_2},\ p_s\nmid b_{i_1} \text{ and } (a_{i_1}a_{i_2}, b_{i_1}b_{i_2})=(p_t\ell,p_s k)$$
$$\text{for some } 1\le i_1,i_2\le e\};$$

$$C(\ell,k)=\{\{(a_1,b_1),(a_2,b_2)\} \mid (1)\ p_t\nmid a_2, a_2\ge 2,\ (2)\ p_s\nmid b_1, b_1\ge 2,$$
$$(3)\ (a_1a_2, b_1b_2)=(p_t\ell,p_s k)\}.$$

Since

$$S=\bigcup_{\substack{\ell\mid m'\\ k\mid n'}}(A(\ell,k)\cup B(\ell,k)),$$

we get the following inequality:

$$g(m,n)=|S|\le\sum_{\substack{\ell\mid m'\\ k\mid n'}}(|A(\ell,k)|+|B(\ell,k)|)\le\sum_{\substack{\ell\mid m'\\ k\mid n'}}(|A(\ell,k)|+|A(\ell,k)|\cdot|C(\ell,k)|)$$
$$\le\sum_{\substack{\ell\mid m'\\ k\mid n'}} g\left(\frac{m'}{\ell},\frac{n'}{k}\right)\{1+(d^{(t)}(\ell)-1)(d^{(s)}(k)-1)\}.$$

From Lemma 2 and the induction hypothesis, we have

$$g(m,n)\le\sum_{\substack{\ell\mid m'\\ k\mid n'}}\frac{h(m')h(n')}{h(\ell)h(k)}\{(d^{(t)}(\ell)-1)(d^{(s)}(k)-1)+1\}=$$

$$= h(m')h(n')\sum_{\substack{\ell|m'\\k|n'}} \frac{d^{(t)}(\ell)d^{(s)}(k) - d^{(s)}(k)d^{(1)}(\ell) - d^{(t)}(\ell)d^{(1)}(k) + 2d^{(1)}(\ell)d^{(1)}(k)}{h(\ell)h(k)}$$

$$\leq \frac{h(m)}{r_t - 1}\frac{h(n)}{r_s - 1}(x^2y^2 - xy^2 - yx^2 + 2xy)$$

$$= h(m)h(n)\frac{(x-1)(y-1)+1}{\sqrt{1+(x-1)^2}\sqrt{1+(y-1)^2}} \leq h(m)h(n),$$

where $x = \prod_{i=1}^{t-1} r_i/(r_i - 1)$ and $y = \prod_{j=1}^{s-1} r_j/(r_j - 1)$. Q.E.D.

Lemma 3: If $m \in M$, $\lambda = 1.143$, then $h(m) \leq m^\lambda$.

Proof: From Lemma 1, we know that $r_i \leq 2.5i$ for all $i \geq 1$. Since λ satisfies the following two inequalities,

(a) $$\left(\prod_{i=1}^{s} p_i\right)^\lambda \geq \prod_{i=1}^{s} r_i \text{ for all } 1 \leq s \leq 12,$$

(b) $$p_i^\lambda \geq (i\log(i))^\lambda \geq i \cdot 12^{\lambda-1}(\log(12))^\lambda \geq 2.5i \geq r_i \text{ for all } i \geq 12,$$

we get $h\left(\prod_{i=1}^t p_i\right) \leq \left(\prod_{i=1}^t p_i\right)^\lambda$ for all $t \geq 1$. (**Note:** $p_i \geq i\log i$ for any positive integer i, see [4].)

From the induction hypothesis on $m \in M$, we have

$$h(m) = h\left(\prod_{i=1}^t p_i^{\alpha_i}\right) = h\left(\prod_{i=1}^t p_i\right)h\left(\prod_{i=1}^t p_i^{\alpha_i - 1}\right) \leq \left(\prod_{i=1}^t p_i\right)^\lambda\left(\prod_{i=1}^t p_i^{\alpha_i-1}\right)^\lambda = m^\lambda,$$

where $m = \prod_{i=1}^t p_i^{\alpha_i}$. Q.E.D.

The following corollary is an immediate consequence of Proposition 1 and Lemma 3 above.

Corollary 1: $g(m,n) \leq (mn)^{1.143}$.

Lemma 4: For any positive integer t,

$$\prod_{i=1}^t \frac{r_i^2}{r_i - u_i} \leq 2160\left(\prod_{i=1}^t p_i\right)^\lambda,$$

where $\lambda = 1.143$ and $u_i = T(p_i)$ for $i \geq 1$.

Proof: Direct computation shows the inequality holds for $t \leq 24$. From Lemma 1 and the Appendix, we know that $2i + 7/4 < r_i < 2.5i$ for all $i \geq 25$. Fix $i \geq 25$. Then we have

$$\frac{r_i^2}{r_i - u_i} \leq \frac{(2.5i)^2}{(2i+1.75)-(i+1.75)} = 6.25i \leq 25^{\lambda-1}(\log 25)^\lambda i \leq (i\log i)^\lambda \leq (p_i)^\lambda. \text{ Q.E.D.}$$

In [2], Mattics and Dodd proved that $f_2(a,1) \leq T(a) \leq a$. Using this fact, we prove the following proposition.

Proposition 2: If $m = \prod_{i=1}^{t} p_i^{\alpha_i}$, $n = \prod_{j=1}^{s} p_j^{\beta_j} \in M$, then

$$f_2(m,n) \le \left(\prod_{i=1}^{t} \frac{r_i^{\alpha_i+1} - u_i^{\alpha_i+1}}{r_i - u_i}\right)\left(\prod_{j=1}^{s} \frac{r_j^{\beta_j+1} - u_j^{\beta_j+1}}{r_j - u_j}\right) \le (2160)^2 (mn)^{1.143},$$

where $u_k = T(p_k)$ for $k \ge 1$.

Proof: For any factorization $(a_1, b_1)(a_2, b_2) \cdots (a_e, b_e)$ of (m, n), there exist unique integers ℓ and k such that

$$\ell = \prod_{\substack{1 \le i \le e \\ b_i = 1}} a_i \quad \text{and} \quad k = \prod_{\substack{1 \le i \le e \\ a_i = 1}} b_i .$$

By Proposition 1, we have

$$f_2(m,n) = \sum_{\substack{\ell | m \\ k | n}} g(m/\ell, n/k) f_2(\ell, 1) f_2(1, k) \le \sum_{\substack{\ell | m \\ k | n}} h(m/\ell) h(n/k) T(\ell) T(k)$$

$$= \left(\sum_{\ell | m} h(m/\ell) T(\ell)\right)\left(\sum_{k | n} h(n/k) T(k)\right) = \left(\prod_{i=1}^{t} \sum_{\ell=0}^{\alpha_i} h(p_i^{\ell}) T(p_i^{\alpha_i - \ell})\right)\left(\prod_{j=1}^{s} \sum_{k=0}^{\beta_j} h(p_j^{k}) T(p_j^{\beta_j - k})\right)$$

$$= \left(\prod_{i=1}^{t} \frac{r_i^{\alpha_i+1} - u_i^{\alpha_i+1}}{r_i - u_i}\right)\left(\prod_{j=1}^{s} \frac{r_j^{\beta_j+1} - u_j^{\beta_j+1}}{r_j - u_j}\right) \le \left(\prod_{i=1}^{t} \frac{r_i^{\alpha_i+1}}{r_i - u_i}\right)\left(\prod_{j=1}^{s} \frac{r_j^{\beta_j+1}}{r_j - u_j}\right).$$

By Lemma 3 and Lemma 4,

$$f_2(m,n) \le \left(\prod_{i=1}^{t} \frac{r_i^2}{r_i - u_i}\right)\left(\prod_{j=1}^{s} \frac{r_j^2}{r_j - u_j}\right)\left(\prod_{i=1}^{t} r_i^{\alpha_i - 1}\right)\left(\prod_{j=1}^{s} r_j^{\beta_j - 1}\right) \le (2160)^2 (mn)^{1.143}. \quad \text{Q.E.D.}$$

Theorem 3: For any pair of positive integers m and n,

$$f_2(m,n) < \frac{(mn)^{1.251}}{\log(mn)}$$

for $mn > 10^{83}$.

Proof: From Proposition 2, it is enough to show that the following inequality holds for $mn > 10^{83}$:

$$(2160)^2 (mn)^{1.143} \le \frac{(mn)^{1.251}}{\log(mn)}.$$

Let $f(t) = t^{0.108} - (2160)^2 \log(t)$. Then we have $f(t) \ge 0$ for $t \ge 10^{83}$, since $f(10^{83}) \ge 0$ and $f'(t) \cdot t = 0.108 t^{0.108} - (2160)^2 > 0$ for all $t \ge 10^{83}$. Hence, we have

$$(mn)^{1.251} - (mn)^{1.143} (2160)^2 \log(mn) = (mn)^{1.143} f(mn) \ge 0$$

for $mn \ge 10^{83}$. Q.E.D.

APPENDIX

The following table shows the sequence $\{r_i\}_{i\geq 1}$ used in this paper.

n	1	2	3	4	5	6
r_n	2	3.82843	6.35584	8.80023	11.1791	13.5137
n	7	8	9	10	11	12
r_n	15.8164	18.0947	20.3538	22.5992	24.8273	27.0461

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