SOME INVARIANT AND MINIMUM PROPERTIES OF STIRLING NUMBERS OF THE SECOND KIND

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The Stirling numbers of the second kind $S(n, k)$ have been studied extensively. This note was motivated by the enumeration of pairwise disjoint finite sequences of random natural numbers. The two main results presented in this note demonstrate some invariant and minimum properties of the Stirling numbers of the second kind.

Combinatorial arguments are used to establish these results; hence, it would be helpful to recall that $S(n, k)$ counts the number of ways to partition a set of n elements into k nonempty subsets. The first main result is

Theorem 1: Let $\mathbf{r} = (r_1, \ldots, r_m)$ be an m-tuple of positive integers, and let N be a positive integer. Denote by $f(N; \mathbf{r})$ the number of m pairwise disjoint finite sequences of rolling an N-faced die, in which the i^{th} sequence consists of r_i trials. Then

$$f(N; \mathbf{r}) = \sum_{\substack{t_1, \ldots, t_m \geq 1 \\ t_1 + \cdots + t_m = N}} \frac{N!}{t_j!} t_j^{r_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} S(r_i, t_i)$$

for any j where $1 \leq j \leq m$.

Proof: Assume there are t_i distinct outcomes from the i^{th} sequence, where $i \neq j$, then the j^{th} sequence consists of at most $t_j = N - \sum_{i \neq j} t_i$ distinct outcomes. There are $\binom{N}{t_1, \ldots, t_m}$ ways to select the possible outcomes. For each $i \neq j$, and a fixed set of t_i outcomes, there are $t_i! S(r_i, t_i)$ ways to roll the die. The j^{th} sequence can be formed in $t_j^{r_j}$ ways. Thus, the total number of ways to roll the die in m disjoint sequences is

$$\sum_{\substack{t_1, \ldots, t_m \geq 1 \\ t_1 + \cdots + t_m = N}} \binom{N}{t_1, \ldots, t_m} t_j^{r_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} t_i! S(r_i, t_i) = f(N; \mathbf{r}).$$

This completes the proof of the claim and the theorem. □

The special case of $m = 2$ appeared in [1]. Its solution (see [2]) can be extended easily to provide another proof of Theorem 1 which is similar to the above proof in spirit. The following corollaries are immediate.

Corollary 2: For $m \geq N$, we have

$$f(N; \mathbf{r}) = \begin{cases} N! & \text{if } m = N, \\ 0 & \text{if } m > N. \end{cases}$$

Corollary 3: The probability that m finite sequences $S_1, S_2, \ldots, S_m$ of rolling a fair N-faced die are pairwise disjoint is $f(N; \mathbf{r}) / N^S$, where $\mathbf{r} = (|S_1|, \ldots, |S_m|)$ and $S = \sum_{k=1}^{m} |S_k|$.

Corollary 4: Given a permutation σ of $\{1, 2, \ldots, m\}$, define $\sigma(\mathbf{r})$ to be $(r_{\sigma(1)}, \ldots, r_{\sigma(m)})$. Then $f(N; \mathbf{r}) = f(N; \sigma(\mathbf{r}))$ for all σ.

In the proof of Theorem 1, if we assume there are t_i distinct outcomes from the i^{th}-sequence for each i, we have

Corollary 5: Let $\mathbf{r}$ and $f(N; \mathbf{r})$ be defined as in Theorem 1. Then

$$f(N; \mathbf{r}) = \sum_{\substack{t_1, \ldots, t_m \geq 1 \\ t_1 + \cdots + t_m \leq N}} \frac{N!}{\left(N - \sum_{i=1}^{m} t_i\right)!} \prod_{i=1}^{m} S(r_i, t_i).$$

We note that $f(N; \mathbf{r})$ can be expressed in an interesting form which possesses a nice commutative property. Let $(N)_i$ denote the falling factorial, $N(N-1) \cdots (N-i+1)$. Given a natural number p, define $N(p)$ as an operator with base N and index p that operates on a polynomial $f(N)$ by the rule

$$N(p) * f(N) = \sum_{j=1}^{\min(p, N)} S(p, j)(N)_j f(N-j).$$

Consider $m = 2$. Suppose the first trial consists of j distinct outcomes, then S_1 and S_2 can be formed in $S(r_1, j)(N)_{r_i}$ and $(N - r_1)^{r_2}$ ways, respectively. Thus,

$$f(N; (r_1, r_2)) = N(r_1) * N^{r_2}.$$

The general result follows by induction:

Theorem 6: The value of $f(N; \mathbf{r})$ has the commutative property

$$f(N; \mathbf{r}) = N(r_1) * \cdots * N(r_{m-1}) * N^{r_m} = N(r_{\sigma(1)}) * \cdots * N(r_{\sigma(m-1)}) * N^{r_{\sigma(m)}}$$

for any permutation σ of $\{1, 2, \ldots, m\}$.

Our second main result is

Theorem 7: Let p and q be positive integers such that $p + q = C$ for some constant C. Then, for a fixed positive integer N, the value of

$$f(N; p, q) = \sum_{i=1}^{\min(p, N)} S(p, i) N(N-1) \cdots (N-i+1)(N-i)^q$$

attains its minimum when $|p - q| \leq 1$.

Proof: From the proof of Theorem 1, we know that $f(N; p, q)$ counts the number of ways to choose from $\{1, 2, \ldots, N\}$ two disjoint sequences S_1 and S_2 of length p and q, respectively. Let $v = 1 - 1/N$ represent the probability that some specific number does not turn up in a single toss of a fair N-faced die. Because of Corollary 3, it suffices to study

$$\begin{aligned}\Pr((r \in S_1 \cup S_2) \wedge (r \notin S_1 \cap S_2)) &= \Pr((r \in S_1) \wedge (r \notin S_2)) + \Pr((r \in S_2) \wedge (r \notin S_1)) \\ &= (1-v^p)v^q + (1-v^q)v^p \\ &= v^p + v^q - 2v^C,\end{aligned}$$

which is minimum if $v^p + v^q$ is. Since this is the sum of two positive numbers whose product v^C is a constant, it follows that it is minimum when $|p-q| \leq 1$. □

Not surprisingly, Theorem 7 can be generalized. Let $S = \sum_{k=1}^m r_k$. Then

$$\begin{aligned}\Pr\left(\left(r \in \bigcup_{k=1}^m S_k\right) \wedge (r \notin S_i \cap S_j,\ 1 \leq i < j \leq m)\right) &= \sum_{k=1}^m (1-v^{r_k})v^{S-r_k} \\ &= v^S \sum_{k=1}^m 1/v^{r_k} - mv^S.\end{aligned}$$

For fixed S, N, m, this probability will be minimum if $L = \sum_{k=1}^m 1/v^{r_k}$ is minimum. But L is the sum of positive numbers whose product $1/v^S$ is a constant; therefore, minimum probability is obtained when the r_i are as nearly equal as possible. In other words, we have $|r_i - r_j| \leq 1$ for any distinct pair of integers i and j. This is equivalent to saying that R of the values r_i equal $Q+1$ and the remaining $m-R$ values equal Q, where $S = mQ + R$, $0 \leq R < m$. This is the same condition under which $f(N; \mathbf{r})$ is minimum.

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