A NOTE ON GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

We let F_n represent the n^{th} *Fibonacci number*. In [2] and [3] we find relationships between the Fibonacci numbers and their associated matrices. The purpose of this paper is to develop relationships between the generalized Fibonacci numbers and the permanent of a (0, 1)-matrix. The *k-generalized Fibonacci sequence* $\{\mathbf{g}_n^{(k)}\}$ is defined as: $\mathbf{g}_1^{(k)} = \mathbf{g}_2^{(k)} = \cdots = \mathbf{g}_{k-2}^{(k)} = 0$, $\mathbf{g}_{k-1}^{(k)} = \mathbf{g}_k^{(k)} = 1$, and, for $n > k \geq 2$,

$$\mathbf{g}_n^{(k)} = \mathbf{g}_{n-1}^{(k)} + \mathbf{g}_{n-2}^{(k)} + \cdots + \mathbf{g}_{n-k}^{(k)}. \tag{1.1}$$

We call $\mathbf{g}_n^{(k)}$ the n^{th} *k-generalized Fibonacci number.*

For example, if $k = 8$, then $\mathbf{g}_1^{(8)} = \cdots = \mathbf{g}_6^{(8)} = 0$, $\mathbf{g}_7^{(8)} = \mathbf{g}_8^{(8)} = 1$, and the sequence of 8-generalized Fibonacci numbers is given by $0, 0, 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, \ldots$.

When $k = 3$, the fundamental recurrence relation $\mathbf{g}_{n+1}^{(3)} = \mathbf{g}_n^{(3)} + \mathbf{g}_{n-1}^{(3)} + \mathbf{g}_{n-2}^{(3)}$ can also be defined by the vector recurrence relation

$$\begin{pmatrix} \mathbf{g}_{n-1}^{(3)} \\ \mathbf{g}_n^{(3)} \\ \mathbf{g}_{n+1}^{(3)} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{g}_{n-2}^{(3)} \\ \mathbf{g}_{n-1}^{(3)} \\ \mathbf{g}_n^{(3)} \end{pmatrix}. \tag{1.2}$$

Letting

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{1.3}$$

and applying (1.2) n times, we have

$$\begin{pmatrix} \mathbf{g}_{n+1}^{(3)} \\ \mathbf{g}_{n+2}^{(3)} \\ \mathbf{g}_{n+3}^{(3)} \end{pmatrix} = A^n \begin{pmatrix} \mathbf{g}_1^{(3)} \\ \mathbf{g}_2^{(3)} \\ \mathbf{g}_3^{(3)} \end{pmatrix}. \tag{1.4}$$

Similarly, for the k-generalized sequence

$$\mathbf{g}_{n+1}^{(k)} = \mathbf{g}_n^{(k)} + \mathbf{g}_{n-1}^{(k)} + \cdots + \mathbf{g}_{n-k+1}^{(k)}, \tag{1.5}$$

the matrix and the vector recurrence relation are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}_{k \times k},$$

and

$$\begin{pmatrix} \mathbf{g}_{n+1}^{(k)} \\ \mathbf{g}_{n+2}^{(k)} \\ \vdots \\ \mathbf{g}_{n+k}^{(k)} \end{pmatrix} = A^n \begin{pmatrix} \mathbf{g}_{1}^{(k)} \\ \mathbf{g}_{2}^{(k)} \\ \vdots \\ \mathbf{g}_{k}^{(k)} \end{pmatrix}. \tag{1.6}$$

We now consider the relationship between $\mathbf{g}_n^{(k)}$ and the *permanent* of a $(0,1)$-matrix. The *permanent* of an n-square matrix $A=[a_{ij}]$ is defined by

$$\operatorname{per} A = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1.7}$$

where the summation extends over all permutations σ of the symmetric group S_n. A matrix is said to be a $(0,1)$-matrix if each of its entries is either 0 or 1.

Let $A=[a_{ij}]$ be an $m\times n$ real matrix with row vectors $\alpha_1,\alpha_2,\dots,\alpha_m$. We say *A is contractible on column* (resp. *row*) *k* if column (resp. *row*) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik}\neq 0\neq a_{jk}$ and $i\neq j$. Then the $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i+a_{ik}\alpha_j$ and deleting row j and column k is called *the contraction of A on column k relative to rows i and j*. If A is contractible on row k with $a_{ki}\neq 0\neq a_{kj}$ and $i\neq j$, then the matrix $A_{k:ij}=[A_{ij:k}^T]^T$ is called *the contraction of A on row k relative to columns i and j*.

We say that *A can be contracted to a matrix B* if either $B=A$ or there exist matrices $A_0, A_1, \dots, A_t$ $(t\geq 1)$ such that $A_0=A$, $A_t=B$, and A_r is a contraction of A_{r-1} for $r=1,\dots,t$.

2. *k*-GENERALIZED FIBONACCI NUMBERS

In [1], we find the following result.

Lemma 1: Let A be a nonnegative integral matrix of order $n>1$ and let B be a contraction of A. Then

$$\operatorname{per} A = \operatorname{per} B. \tag{2.1}$$

Furthermore, if we let $\mathcal{F}^{(n,k)}=[f_{ij}]$ be the $n\times n$ $(0,1)$-$(k+1)^{\text{st}}$ (*super diagonal*) *matrix* defined by

$$\mathcal{F}^{(n,k)} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & & & & & & \ddots & \vdots \\ & & \ddots & \ddots & & & & & \ddots & 0 \\ \vdots & & & \ddots & \ddots & & \ddots & & & 1 \\ \vdots & & & & \ddots & \ddots & & & & \vdots \\ 0 & & & \cdots & \cdots & & 0 & 1 & & 1 \end{bmatrix}, \tag{2.2}$$

where $f_{11}=\cdots=f_{1k}=1$ and $f_{1k+1}=\cdots=f_{1n}=0$, then $\mathcal{F}^{(n,k)}$ is contractible on column 1 relative to rows 1 and 2. In particular, if $k=2$, then $\mathcal{F}^{(n,k)}$ is turned to be the $(0,1)$-tridiagonal (*Toeplitz*) matrix $T^{(n)}$ of order n.

Lemma 2: Let $T_p^{(n)} = [t_{ij}]$ be the p^{th} contraction of the matrix $T^{(n)}$, $1 \le p \le n-2$. Then $t_{11} = F_{p+2}$ and $t_{12} = F_{p+1}$, where F_p is the p^{th} *Fibonacci number*, $p = 1, 2, \ldots, n-2$.

Proof: We use induction on p. Since

$$T_1^{(n)} = \begin{bmatrix} 2 & 1 & & 0 & \\ 1 & 1 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ & 0 & & 1 & 1 \end{bmatrix},$$

the case for $p = 1$ is true. Since

$$T_{p-1}^{(n)} = \begin{bmatrix} F_{p+1} & F_p & & & 0 \\ 1 & 1 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ & 0 & & 1 & 1 \end{bmatrix},$$

by the induction assumption, $T_{p-1}^{(n)}$ is contractible on column 1 relative to rows 1 and 2. Thus,

$$T_p^{(n)} = \begin{bmatrix} F_{p+1}+F_p & F_{p+1} & & 0 & \\ 1 & 1 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ & 0 & & 1 & 1 \end{bmatrix}.$$

However, $F_{p+1} + F_p = F_{p+2}$, $t_{11} = F_{p+2}$ and $t_{12} = F_{p+1}$, so the proof is complete.

Lemma 3: Let $\mathscr{F}_t^{(n,k)} = [f_{ij}]$ be the t^{th} contraction of $\mathscr{F}^{(n,k)}$, $1 \le t \le n-2$. Then, for $k > t+1$,

$$\begin{aligned} f_{11} = \cdots f_{1k-t} &= \mathbf{g}_{k+t}^{(k)}, \\ f_{1j} &= f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)}, \end{aligned} \quad k-t+1 \le j \le n-t,$$

and, for $k \le t+1$,

$$\begin{aligned} f_{11} &= g_{k+t}, \\ f_{1j} &= f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)}, \end{aligned} \quad 2 \le j \le n-t.$$

In any case, if $f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)} < 0$, we let f_{ij} be zero.

Proof: We proceed by induction. The result is easily established for $t = 1$. We now assume the theorem is true for t and consider $\mathscr{F}_{t+1}^{(n,k)}$. We examine two cases.

For the first case, assume $k > t+1$. Let $\mathscr{F}_t^{(n,k)} = [f_{ij}]$. Then $f_{11} = \cdots = f_{1k-t} = \mathbf{g}_{k+t}^{(k)}$ and $f_{1j} = f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)}$, $k-t+1 \le j \le n-t$. Let $\mathscr{F}_{t+1}^{(n,k)} = [f_{ij}^{\dagger}]$. By contradiction,

$$\begin{aligned} f_{1q}^{\dagger} &= f_{11} + f_{1p},\ p = 2, \ldots, k = t,\ q = 1, \ldots, k - t - 1 \\ &= \mathbf{g}_{k+t}^{(k)} + \mathbf{g}_{k+t}^{(k)} \\ &= \mathbf{g}_{k+t}^{(k)} + \mathbf{g}_{k+t-1}^{(k)} + \cdots + \mathbf{g}_{t}^{(k)}. \end{aligned}$$

Since $k > t+1$, $\mathbf{g}_t^{(k)} = 0$. Thus, $f_{1q}^{\dagger} = \mathbf{g}_{k+t+1}^{(k)}$, $q = 1, \ldots, k-(t+1)$, and

$$\begin{aligned} f_{1k+t}^{\dagger} &= f_{11} + f_{1k-t+1} \\ &= f_{11} + f_{1k-t} - \mathbf{g}_{t+k-t+1-2}^{(k)} \\ &= f_{11} + f_{1k-t} - \mathbf{g}_{k-1}^{(k)} \\ &= f_{1k-t-1}^{\dagger} - \mathbf{g}_{(t+1)+(k-t)-2}^{(k)}. \end{aligned}$$

Hence, $f_{1k-t}^{\dagger} = f_{1k-t-1}^{\dagger} - \mathbf{g}_{(t+1)+(k-t)-2}^{(k)}$. So, by the recurrence relation

$$f_{1j}^{\dagger} = f_{1j-1}^{\dagger} - \mathbf{g}_{(t+1)+j-2}^{(k)},\ k - t \le j \le n-(t-1).$$

For the second case, we let $k \le t+1$. If $t = 1$, then $k = 2$ and we are done, by Lemma 2. Let $\mathcal{F}_t^{(n,k)} = [f_{ij}]$. Then $f_{11} = \mathbf{g}_{k+t}^{(k)}$ and $f_{1j} = f_{1j-1} - \mathbf{g}_{t+j-2}^{(k)}$, $2 \le j \le n-t$. Let $\mathcal{F}_{t+1}^{(n,k)} = [f_{ij}^{\dagger}]$. Then, by Lemma 1,

$$\begin{aligned} f_{11}^{\dagger} &= f_{11} + f_{12} \\ &= \mathbf{g}_{k+t}^{(k)} + f_{11} - \mathbf{g}_{t+2-2}^{(k)} \\ &= \mathbf{g}_{k+t}^{(k)} + \mathbf{g}_{k+t}^{(k)} - \mathbf{g}_{t}^{(k)} \\ &= \mathbf{g}_{k+t}^{(k)} + \mathbf{g}_{k+t-1}^{(k)} + \cdots + \mathbf{g}_{t}^{(k)} - \mathbf{g}_{t}^{(k)} \\ &= \mathbf{g}_{k+t}^{(k)} + \cdots + \mathbf{g}_{t+1}^{(k)} \\ &= \mathbf{g}_{k+t+1}, \end{aligned} \qquad \begin{aligned} f_{12}^{\dagger} &= f_{11} + f_{13} \\ &= f_{11} + f_{12} - \mathbf{g}_{t3-2}^{(k)} \\ &= f_{11} + (f_{11} - \mathbf{g}_{t}^{(k)}) - \mathbf{g}_{t+1}^{(k)} \\ &= \mathbf{g}_{k+t+1}^{(k)} - \mathbf{g}_{t+1}^{(k)} \\ &= f_{11}^{\dagger} - \mathbf{g}_{(t+1)+2-2}^{(k)}, \end{aligned}$$

so that $f_{12}^{\dagger} = f_{11}^{\dagger} - \mathbf{g}_{(t+1)+2-2}^{(k)}$. Thus, by the recurrence relation, $f_{ij}^{\dagger} = f_{1j-1}^{\dagger} - \mathbf{g}_{(t+1)+j-2}^{(k)}$ and the proof is completed.

Theorem 1: Let $\mathbf{g}_{n+1}^{(k)}$ be the $(n+1)^{\text{st}}$ k-generalized Fibonacci number, $n \ge k$. Then

$$\operatorname{per} \mathcal{F}^{(n,k)} = \mathbf{g}_{n+k-1}^{(k)}. \tag{2.3}$$

Proof: Since $\mathcal{F}^{(n,k)}$ is contractible, $\mathcal{F}^{(n,k)}$ can be contracted to a 2-square integral matrix B. By Lemma 3,

$$B = \mathcal{F}_{n-2}^{(n,k)} = \begin{bmatrix} \mathbf{g}_{n+k-2}^{(k)} & \mathbf{g}_{n+k-2}^{(k)} - \mathbf{g}_{n-2}^{(k)} \\ 1 & 1 \end{bmatrix},$$

and by Lemma 1,

$$\begin{aligned} \operatorname{per} \mathcal{F}^{(n,k)} = \operatorname{per} B &= \mathbf{g}_{n+k-2}^{(k)} + \mathbf{g}_{n+k-2}^{(k)} - \mathbf{g}_{n-2}^{(k)} \\ &= \mathbf{g}_{n+k-2}^{(k)} + \mathbf{g}_{n+k-3}^{(k)} + \cdots + \mathbf{g}_{n-2}^{(k)} - \mathbf{g}_{n-2}^{(k)} \\ &= \mathbf{g}_{n+k-2}^{(k)} + \mathbf{g}_{n+k-3}^{(k)} + \cdots + \mathbf{g}_{n-1}^{(k)} \\ &= \mathbf{g}_{n+k-1}^{(k)}, \end{aligned}$$

and the proof is completed.

Corollary: The $(n+1)^{\text{st}}$ Fibonacci number is equal to the permanent of the $(0,1)$-tridiagonal matrix of order n.

The next theorem shows that we can find a nontridiagonal matrix whose permanent also equals the $(n+1)^{\text{st}}$ Fibonacci number.

Theorem 2: Let

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & \cdots & & 0 & 1 & 1 \end{bmatrix}_{n\times n}.$$

Then

$$\operatorname{per} P^T U P = F_{n+1}. \tag{2.4}$$

for any permutation matrix P.

Proof: The matrix U can be contracted on column 1 so that

$$U_1 = \begin{bmatrix} 1 & 2 & 2 & 2 & \cdots & 2 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & \cdots & & 1 & 1 & 1 \end{bmatrix},$$

where $(U_1)_{11} = 1 = F_2$ and $(U_1)_{12} = 2 = F_3$. Furthermore, the matrix U_1 can be contracted on column 1 so that

$$U_2 = \begin{bmatrix} 2 & 3 & 3 & 3 & \cdots & 3 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & \cdots & & 1 & 1 & 1 \end{bmatrix},$$

where $(U_2)_{11} = 2 = F_3$ and $(U_2)_{12} = 3 = F_4$. Continuing this process, we have

$$U_t = \begin{bmatrix} F_{t+1} & F_{t+2} & F_{t+2} & F_{t+2} & \cdots & F_{t+2} \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & \cdots & & 0 & 1 & 1 \end{bmatrix}$$

for $1 \le t \le n-2$. Hence,

$$U_{n+3} = \begin{bmatrix} F_{n+2} & F_{n-1} & F_{n-1} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

which, by contraction of U_{n-3} on column 1, gives

$$U_{n-2} = \begin{bmatrix} F_{n-1} & F_{n-1}+F_{n-2} \\ 1 & 1 \end{bmatrix} = U_{n-2} = \begin{bmatrix} F_{n-1} & F_n \\ 1 & 1 \end{bmatrix}.$$

Applying Lemma 1, we have

$$\text{per } U = \text{per } U_t = \text{per } U_{n-2} = F_n + F_{n-1} = F_{n+1}.$$

Since the permanent is permutation similarity invariant, the proof is completed.

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