ON RECIPROCAL SUMS OF CHEBYSHEV RELATED SEQUENCES

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1. INTRODUCTION

Define the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ for any real number p by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0, \ U_1 = 1, \ n \ge 2, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2, \ V_1 = p, \ n \ge 2. \end{cases} \tag{1.1}$$

They can be extended to negative subscripts by the use of the recurrence relation. The Binet forms are

$$U_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \tag{1.2}$$

$$V_n = \gamma^n + \delta^n, \tag{1.3}$$

where

$$\gamma = \frac{p + \sqrt{p^2+4}}{2}, \quad \delta = \frac{p - \sqrt{p^2+4}}{2}. \tag{1.4}$$

These sequences are generalizations of the Fibonacci and Lucas sequences and as such have appeared frequently in the literature. See, for example, [5], [9], [10], [19], and [24]. Lucas [23] studied the following generalizations of the sequences (1.1):

$$U_n = pU_{n-1} - qU_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \quad n \ge 2,$$
$$V_n = pV_{n-1} - qV_{n-2}, \quad V_0 = 2, \quad V_1 = p, \quad n \ge 2.$$

Let $\{T_n(x)\}_{n=0}^{\infty}$ and $\{S_n(x)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$S_n(x) = \sin n\theta / \sin\theta, \ x = \cos\theta, \ n \ge 0, \tag{1.5}$$

$$T_n(x) = \cos n\theta, \ x = \cos\theta, \ n \ge 0. \tag{1.6}$$

These polynomial sequences have been studied extensively. See, for example, [1], [11], and [22]. Lucas states (see [23], p. 189) that Jean Bernoulli, in 1701, expressed $\sin n\theta / \sin\theta$ and $\cos n\theta$ in powers of $\sin\theta$ and $\cos\theta$. Lucas also states (p. 208) that Viète (sometimes spelled Vièta), in 1646, was the first to express $\sin n\theta / \sin\theta$ and $\cos n\theta$ as sums of powers of $\cos\theta$. Chebyshev's contributions came much later since he lived during the years 1821-1894. Lucas also quotes (p. 195) a continued fraction expansion for $\sin(n+1)\theta / \sin n\theta$ in which $2\cos\theta$ is repeated n times.

Writing $U_n = S_n(x)$, $V_n = 2T_n(x)$, $p = 2x$, we have the familiar recurrences (see [20], [22]),

$$\begin{cases} U_n = pU_{n-1} - U_{n-2}, & U_0 = 0, \ U_1 = 1, \ n \ge 2, \\ V_n = pV_{n-1} - V_{n-2}, & V_0 = 2, \ V_1 = p, \ n \ge 2. \end{cases} \tag{1.7}$$

Because of their connection with the Chebyshev polynomials, the sequences (1.7) merit closer attention. We have studied them independently of the Chebyshev polynomials, taking p to be a real number. In particular, we have noticed that many properties of the sequences (1.1) have interesting analogs for the sequences (1.7).

In this paper we focus on results for the sequences (1.1) involving certain reciprocal sums and obtain analogs for the sequences (1.7).

Unless otherwise stated, we assume throughout that in (1.7) p is real and $|p|>2$. The Binet forms for the sequences (1.7) are then

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.8}$$

$$V_n = \alpha^n + \beta^n, \tag{1.9}$$

where

$$\alpha = \frac{p+\sqrt{p^2-4}}{2}, \quad \beta = \frac{p-\sqrt{p^2-4}}{2}. \tag{1.10}$$

We use the $U_n - V_n$ notation throughout to refer to the sequences (1.1) and to the sequences (1.7). There will be no ambiguity since we shall always indicate the set to which we are referring.

2. THE RESULTS OF GOOD AND GRIEG

Good [12] showed that

$$\sum_{i=0}^{n} \frac{1}{F_{2^i}} = 3 - \frac{F_{2^n-1}}{F_{2^n}}, \quad n \geq 1, \tag{2.1}$$

from which he obtained the corresponding infinite sum by noting that $\frac{F_{n-1}}{F_n} \to \frac{\sqrt{5}-1}{2}$.

In a comprehensive paper, Hoggatt and Bicknell [16] indicate eleven methods of obtaining the corresponding infinite sum, highlighting the contributions of various authors. Bruckman and Good [8] give several interesting sums involving reciprocals. They state that Hoggatt, in an unpublished note (December 1974) obtained the sum

$$\sum_{i=1}^{\infty} \frac{1}{F_{k2^i}}.$$

Later Hoggatt and Bicknell [17] generalized this by evaluating

$$\sum_{i=0}^{n} \frac{1}{F_{k2^i}}.$$

Greig [15] gave a different version of the last sum by showing that

$$\sum_{i=0}^{n} \frac{1}{F_{k2^i}} = C_k - \frac{F_{k2^n-1}}{F_{k2^n}}, \quad n, k \geq 1, \tag{2.2}$$

where C_k is independent of n and is given by

$$C_k = \begin{cases} \dfrac{1+F_{k-1}}{F_k}, & k \text{ even}, \\ \dfrac{1+F_{k-1}}{F_k} + \dfrac{2}{F_{2k}}, & k \text{ odd}. \end{cases} \tag{2.3}$$

By using a certain partition of the natural numbers, Greig obtained

$$\sum_{i=1}^{\infty} \frac{1}{F_i} = \sum_{k=0}^{\infty} \left(C_{2k+1} - \frac{1}{\phi} \right), \quad \phi = \frac{1+\sqrt{5}}{2}, \tag{2.4}$$

and other variant forms. He did this by observing that, as k and m take on all integer values such that $k \geq 0$ and $m \geq 0$, $(2k+1)2^m$ generates each natural number once. Then he used the following rearrangement theorem in conjunction with (2.2) and (2.3) to obtain (2.4) and its variants.

Theorem 1:

$$\sum_{i=1}^{\infty} f(i) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f((2k+1)2^n)$$

for an arbitrary function f provided only that the series on the left converges absolutely.

Gould [13], commenting on Greig's paper, remarked that Theorem 1 "seems to be common knowledge in the mathematical community, but its use in forming interesting series rearrangements does not seem to be widely known or appreciated." He then used Theorem 1 to prove that, for $|x| < 1$,

$$\sum_{i=0}^{\infty} \frac{x^{2^i}}{1 - x^{2^{i+1}}} = \frac{x}{1-x}.$$

Indeed, Bromwich (see [7], p. 24) attributed this formula to De Morgan. The paper by Bruckman and Good [8] was based on generalizations of this formula. Gould then stated a generalized version of Theorem 1 and gave interesting applications to summations involving the Riemann zeta function, Euler's ϕ-function, and the Trigonometric and Hyperbolic functions. We will use Theorem 1 subsequently.

In a further paper, Greig [14] obtained formulas analogous to (2.1)-(2.4) for the sequence $\{U_n\}$ given in (1.1). Gould [13] commented that similar results for L_n seem to be unattainable using the method of Greig. Horadam [18] made the same comment in relation to the Pell-Lucas numbers (i.e., the sequence $\{V_n\}$ in (1.1) with $p = 2$). Badea [4] proved that

$$\sum_{i=0}^{\infty} \frac{1}{L_{2^i}}$$

is irrational, and recently André-Jeannin [2] proved that

$$\sum_{i=1}^{\infty} \frac{\varepsilon^i}{L_{2^i}},$$

where $\varepsilon = \pm 1$, does not belong to $Q(\sqrt{5})$. These results suggest that a sum corresponding to (2.1) for L_n does not exist.

We now obtain results analogous to (2.1)-(2.4) for the sequence $\{U_n\}$ given in (1.7), where, as stated earlier, we assume that $|p|>2$.

Theorem 2:

$$\sum_{i=0}^{n}\frac{1}{U_{2^i}}=1+\frac{U_{2^n-1}}{U_{2^n}}.$$

Now, since U_{n-1}/U_n approaches $\frac{1}{\alpha}(p>2)$ or $\frac{1}{\beta}(p<-2)$, we have as a corollary of Theorem 2

Theorem 3:

$$\sum_{i=0}^{\infty}\frac{1}{U_{2^i}}=\begin{cases}1+\dfrac{1}{\alpha}, & p>2,\\[2mm] 1+\dfrac{1}{\beta}, & p<-2.\end{cases}$$

We now give a generalization of Theorem 2. The proof relies on the following result which can be established using Binet forms:

$$U_{2j-1}U_j-U_{2j}U_{j-1}=U_j. \tag{2.5}$$

We mention in passing that the corresponding identity for the Fibonacci numbers is

$$F_{2j-1}F_j-F_{2j}F_{j-1}=(-1)^{j-1}F_j.$$

Theorem 4: For $k\geq 1$ an integer,

$$\sum_{i=0}^{n}\frac{1}{U_{k2^i}}=\frac{1-U_{k-1}}{U_k}+\frac{U_{k2^n-1}}{U_{k2^n}}.$$

Proof: We proceed by induction. When $n=0$, both sides reduce to $1/U_k$. The inductive step requires us to prove that

$$U_{k2^{n+1}-1}U_{k2^n}-U_{k2^{n+1}}U_{k2^n-1}=U_{k2^n}$$

This is achieved by putting $j=k2^n$ in (2.5) and the proof of Theorem 4 is complete. □

As a corollary, we have

Theorem 5: For $k\geq 1$ an integer,

$$\sum_{i=0}^{\infty}\frac{1}{U_{k2^i}}=\begin{cases}\dfrac{1-U_{k-1}}{U_k}+\dfrac{1}{\alpha}, & p>2,\\[2mm] \dfrac{1-U_{k-1}}{U_k}+\dfrac{1}{\beta}, & p<-2.\end{cases}$$

We now use Theorem 1 to obtain an interesting bisection result for the sum of the reciprocals of $\{U_n\}_{n=1}^{\infty}$. We shall need the following result which is easily established using Binet forms:

$$\frac{1}{\alpha}-\frac{U_{k-1}}{U_k}=\frac{1}{\alpha^k U_k},\quad k\geq 1. \tag{2.6}$$

Using Theorem 1 we have, for $p > 2$,

$$\sum_{i=1}^{\infty} \frac{1}{U_i} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{U_{(2k+1)2^n}}$$
$$= \sum_{k=0}^{\infty} \left(\frac{1}{U_{2k+1}} + \frac{1}{\alpha} - \frac{U_{2k}}{U_{2k+1}} \right) \quad \text{(by Theorem 5)}$$
$$= \sum_{k=0}^{\infty} \left(\frac{1}{U_{2k+1}} + \frac{1}{\alpha^{2k+1} U_{2k+1}} \right) \quad \text{[by (2.6)]}$$

and so

$$\sum_{i=1}^{\infty} \frac{1}{U_i} = \sum_{i=0}^{\infty} \frac{1}{U_{2i+1}} + \sum_{i=0}^{\infty} \frac{1}{\alpha^{2i+1} U_{2i+1}}. \tag{2.7}$$

Comparing both sides of (2.7) leads to

$$\sum_{i=1}^{\infty} \frac{1}{U_{2i}} = \sum_{i=0}^{\infty} \frac{1}{\alpha^{2i+1} U_{2i+1}}. \tag{2.8}$$

For $p < -2$, simply replace α by β.

Numerical examples suggest that the right side of (2.8) converges much faster than the left side. For example, taking $p = 3$ gives $U_n = F_{2n}$, and (2.8) becomes

$$\sum_{i=1}^{\infty} \frac{1}{F_{4i}} = \sum_{i=0}^{\infty} \frac{1}{\alpha^{2i+1} F_{4i+2}}, \tag{2.9}$$

where $\alpha = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$.

Taking twelve terms on the left side gives the value 0.389083066... but only six terms on the right are needed to achieve this value.

Note: Since $|p| > 2$, use of the ratio test shows that the left side of (2.7) is absolutely convergent, so that the use of Theorem 1 is valid.

3. THE RESULTS OF ANDRÉ-JEANNIN

We now examine reciprocal sums of a different nature. For the sequences (1.1), André-Jeannin [3] proved the following.

Theorem 6: If k is an odd integer and $p > 0$, then

$$\sum_{i=1}^{\infty} \frac{1}{U_{ki} U_{k(i+1)}} = \frac{2(\gamma - \delta)}{U_k} [L(\delta^{2k}) - 2L(\delta^{4k}) + 2L(\delta^{8k})] + \frac{\delta^k}{U_k^2},$$
$$\sum_{i=1}^{\infty} \frac{1}{V_{ki} V_{k(i+1)}} = \frac{2}{(\gamma - \delta) U_k} [L(\delta^{2k}) - 2L(\delta^{8k})] + \frac{\delta^k}{(\gamma - \delta) U_k V_k}.$$

Here, $L(x)$ is the Lambert series defined by

$$L(x)=\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}, \quad |x|<1.$$

Information about the Lambert series can be found in Knopp [21] and in Borwein and Borwein [6]. In the latter reference there are numerous reciprocal sums for the Fibonacci and Lucas numbers involving the Lambert series and the theta functions of Jacobi. On pages 94-95 in [6], the sums $\sum_{n=0}^{\infty}F_{2n+1}^{-1}$ and $\sum_{n=1}^{\infty}F_{2n}^{-1}$, due to Landau, are stated.

We now obtain sums analogous to those in Theorem 6 for the sequences (1.7). Interestingly, these sums do not involve $L(x)$, and the requirement that k be odd is not needed. For the remainder of this section, U_n and V_n are as in (1.7). We shall need the following lemma.

Lemma 1: For integers k and n,

$$\alpha^k U_{k(n+1)}-U_{kn}=\alpha^{k(n+1)}U_k, \tag{3.1}$$

$$\alpha^k V_{k(n+1)}-V_{kn}=(\alpha-\beta)\alpha^{k(n+1)}U_k. \tag{3.2}$$

Proof: We prove only (3.2) since the proof of (3.1) is similar. Recalling that $\alpha\beta=1$, we have

$$\begin{aligned}\alpha^k V_{k(n+1)}-V_{kn}&=\alpha^k(\alpha^{k(n+1)}+\beta^{k(n+1)})-(\alpha^{kn}+\beta^{kn})\\&=\alpha^{kn+2k}-\alpha^{kn}\\&=\alpha^{kn}(\alpha^{2k}-\alpha^k\beta^k)\\&=\alpha^{k(n+1)}(\alpha^k-\beta^k)\\&=(\alpha-\beta)\alpha^{k(n+1)}U_k.\end{aligned}$$

Before proceeding we note that, for $|p|\geq 2$, $\{|U_n|\}_{n=0}^{\infty}$ is an increasing sequence and, for $|p|>2$, $\{|V_n|\}_{n=0}^{\infty}$ is an increasing sequence.

Theorem 7: For $|p|\geq 2$, $k\neq 0$ an integer,

$$\sum_{i=1}^{n}\frac{1}{U_{ki}U_{k(i+1)}}=\frac{1}{\alpha^k U_k}\left[\frac{1}{U_k}-\frac{1}{\alpha^{kn}U_{k(n+1)}}\right], \tag{3.3}$$

$$\sum_{n=1}^{\infty}\frac{1}{U_{ki}U_{k(i+1)}}=\frac{1}{\alpha^k U_k^2}. \tag{3.4}$$

Proof:

$$\begin{aligned}\frac{1}{\alpha^{ki}U_{ki}}-\frac{1}{\alpha^{k(i+1)}U_{k(i+1)}}&=\frac{\alpha^k U_{k(i+1)}-U_{ki}}{\alpha^{k(i+1)}U_{ki}U_{k(i+1)}}\\&=\frac{U_k}{u_{ki}U_{k(i+1)}} \quad \text{[by (3.1)]}.\end{aligned}$$

Letting $i = 1, 2, \ldots, n$ and summing both sides proves (3.3). Letting $n \to \infty$ in (3.3) establishes (3.4). □

Making use of (3.2) and proceeding in the same manner, we obtain

Theorem 8: $|p| > 2$, $k \neq 0$ an integer,

$$\sum_{i=0}^{n} \frac{1}{V_{ki} V_{k(i+1)}} = \frac{1}{(\alpha - \beta) U_k} \left[\frac{1}{2} - \frac{1}{\alpha^{k(n+1)} V_{k(n+1)}} \right], \tag{3.5}$$

$$\sum_{i=0}^{\infty} \frac{1}{V_{ki} V_{k(i+1)}} = \frac{1}{2(\alpha - \beta) U_k}. \tag{3.6}$$

As an application of our results we note that, when $p = 2$,

$$\{U_n\}_{n=0}^{\infty} = \{0, 1, 2, 3, \ldots\} \text{ and } \alpha = \beta = 1.$$

Although the Binet form for U_n degenerates, (3.1) remains valid, and Theorem 7 yields the familiar sums:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}, \tag{3.7}$$

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1. \tag{3.8}$$

As another application, taking $p = 3$ gives $U_n = F_{2n}$, $V_n = L_{2n}$, and (3.4) and (3.6) become, respectively,

$$\sum_{i=1}^{\infty} \frac{1}{F_{2ki} F_{2k(i+1)}} = \frac{2^k}{(3 + \sqrt{5})^k F_{2k}^2}, \tag{3.9}$$

$$\sum_{i=0}^{\infty} \frac{1}{L_{2ki} L_{2k(i+1)}} = \frac{\sqrt{5}}{10 F_{2k}}. \tag{3.10}$$

4. CONCLUDING COMMENTS

We feel that there is more scope for developing results for the sequences (1.7) which parallel existing results for the sequences (1.1). For example, Brugia, Di Porto, and Filipponi [9] investigated the sum

$$\begin{aligned} S &= \sum_{i=0}^{\infty} \frac{U_i}{r^i}, \quad r \neq 0, \\ &= \frac{r}{r^2 - pr - 1}, \quad |r| > \gamma, \end{aligned}$$

where U_n is as in (1.1). They established the following theorem.

Theorem 9: If U_n is as in (1.1), where p is integral, the rational values of r for which S is integral are given by

$$r=\frac{U_{2n+1}}{U_{2n}} \quad (n=1,2,3,\ldots),$$

and the corresponding value of S is given by

$$S=U_{2n}U_{2n+1} \quad (n=1,2,3,\ldots).$$

We have investigated precisely the same sum for U_n as in (1.7), with $|r|>\alpha$, and have found the following parallel result.

Theorem 10: If U_n is as in (1.7), where $p>2$ is integral, the rational values of r for which S is integral are given by

$$r=\frac{U_{n+1}}{U_n} \quad (n=1,2,3,\ldots),$$

and the corresponding value of S is given by

$$S=U_nU_{n+1} \quad (n=1,2,3,\ldots).$$

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REFERENCES

1. M. Abramowitz & I. A. Stegun. *Handbook of Mathematical Functions*. New York: Dover, 1972.
2. R. André-Jeannin. "A Note on the Irrationality of Certain Lucas Infinite Series." *The Fibonacci Quarterly* **29.2** (1991):132-36.
3. R. André-Jeannin. "Lambert Series and the Summation of Reciprocals in Certain Fibonacci-Lucas-Type Sequences." *The Fibonacci Quarterly* **28.3** (1990):223-26.
4. C. Badea. "The Irrationality of Certain Infinite Series." *The Glasgow Mathematics Journal* **29** (1987):221-28.
5. M. Bicknell. "A Primer on the Pell Sequence and Related Sequences." *The Fibonacci Quarterly* **13.4** (1975):345-48.
6. J. M. Borwein & P. B. Borwein. *Pi and the AGM*, pp. 91-101. New York: Wiley, 1987.
7. T. Bromwich. *An Introduction to the Theory of Infinite Series*, 2nd ed. London: Macmillan, 1926.
8. P. S. Bruckman & I. J. Good. "A Generalization of a Series of De Morgan with Applications of Fibonacci Type." *The Fibonacci Quarterly* **14.3** (1976):193-96.
9. O. Brugia, A. Di Porto, & P. Filipponi. "On Certain Fibonacci-Type Sums." *International Journal of Mathematical Education in Science and Technology* **22** (1991):609-13.
10. E. Cohn. "Pell Number Triples." *The Fibonacci Quarterly* **10.5** (1972):403-12.
11. A. Erdelyi et al. *Higher Transcendental Functions*, Vol. 2. New York: McGraw-Hill, 1953.
12. I. J. Good. "A Reciprocal Series of Fibonacci Numbers." *The Fibonacci Quarterly* **12.4** (1974):346.
13. H. W. Gould. "A Rearrangement of Series Based on a Partition of the Natural Numbers." *The Fibonacci Quarterly* **15.1** (1977):67-72.
14. W. E. Greig. "On Sums of Fibonacci-Type Reciprocals." *The Fibonacci Quarterly* **15.4** (1977):356-58.

15. W. E. Greig. "Sums of Fibonacci Reciprocals." *The Fibonacci Quarterly* **15.1** (1977):46-48.
16. V. E. Hoggatt, Jr., & M. Bicknell. "A Primer for the Fibonacci Numbers, Part XV: Variations on Summing a Series of Reciprocals of Fibonacci Numbers." *The Fibonacci Quarterly* **14.3** (1976):272-76.
17. V. E. Hoggatt, Jr., & M. Bicknell. "A Reciprocal Series of Fibonacci Numbers with Subscripts $2^n k$." *The Fibonacci Quarterly* **14.5** (1976):453-55.
18. A. F. Horadam. "Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences." *The Fibonacci Quarterly* **26.2** (1988):98-114.
19. A. F. Horadam. "Generating Identities for Generalized Fibonacci and Lucas Triples." *The Fibonacci Quarterly* **15.4** (1977):289-93.
20. A. F. Horadam. "Tschebyscheff and Other Functions Associated with the Sequence $\{W_n(a,b; p,q)\}$." *The Fibonacci Quarterly* **7.1** (1969):14-22.
21. K. Knopp. *Theory and Application of Infinite Series*. London?: Blackie, 1949.
22. E. Kreyszig. *Introductory Functional Analysis with Applications*. New York: Wiley, 1978.
23. E. Lucas. "Théorie des fonctions numériques simplement périodiques." *American Journal of Mathematics* **1** (1878):184-240, 289-321.
24. Bro. J. M. Mahon. "Pell Polynomials." M.A. Thesis presented to the University of New England, 1984.
25. D. A. Millin. Problem H-237. *The Fibonacci Quarterly* **12.4** (1974):309.
26. A. G. Shannon. Solution to Problem H-237. *The Fibonacci Quarterly* **14.2** (1976):186-87.

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