NUMBERS OF SUBSEQUENCES WITHOUT ISOLATED ODD MEMBERS

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We find the numbers of subsequences of $\{1, 2, \ldots, n\}$ in which every odd member is accompanied by at least one even neighbor. For example, 123568 is acceptable, but 123578 is not, since 5 has no even neighbor. The empty sequence is always acceptable. Preliminary calculations for $0 \le n \le 8$ yield the following values of $z_{n,c}$, the number of such subsequences of length c. It is convenient to define $z_{n,c} = 0$ if c is not in the interval $0 \le c \le n$ and $z_n = \sum_{c=0}^{n} z_{n,c}$ is the total number of such subsequences. x_n and $x_{n,c}$ are the corresponding numbers of subsequences from which n is **excluded**, whereas n **does** occur in the subsequences counted by y_n and $y_{n,c}$. Of course, $x_n + y_n = z_n$ with similar formulas for specific lengths.

The following tables suggest several simple relations, which are easily verified by considering the last two or three members of the relevant subsequences:

$$\begin{array}{lll} x_{n+1} = z_n, & x_{n+1,c} = z_{n,c} & (n \ge 0), \\ y_{2k+1} = y_{2k}, & y_{2k+1,c+1} = y_{2k,c} & (k \ge 0), \\ y_{2k} = z_{2k-1} + z_{2k-3}, & y_{2k,c+1} = z_{2k-1,c} + z_{2k-3,c-1} & (k \ge 0), \end{array}$$

where we adopt the conventions $z_{-1} = z_{-1,0} = 1$ and $x_{-2} = z_{-3} = z_{-3,-1} = -1$.

TABLE 1. $z_{n,c}$, $c = 0, 1, 2, \ldots, n$

n										z_n
0	1									1
1	1	0								1
2	1	1	1							3
3	1	1	2	1						5
4	1	2	4	3	1					11
5	1	2	5	5	3	1				17
6	1	3	8	11	10	5	1			39
7	1	3	9	14	16	12	5	1		61
8	1	4	13	25	35	33	20	7	1	139

The corresponding arrays for the values of x and y are given in Tables 2 and 3, respectively.

TABLE 2. $x_{n,c}$, $c = 0, 1, 2, \ldots, n$

n										x_n
0	1									1
1	1	0								1
2	1	0	0							1
3	1	1	1	0						3
4	1	1	2	1	0					5
5	1	2	4	3	1	0				11
6	1	2	5	5	3	1	0			17
7	1	3	8	11	10	5	1	0		39
8	1	3	9	14	16	12	5	1	0	61

TABLE 3. $y_{n,c}$, $c = 0, 1, 2, \ldots, n$

n										y_n
0	0									0
1	0	0								0
2	0	1	1							2
3	0	0	1	1						2
4	0	1	2	2	1					6
5	0	0	1	2	2	1				6
6	0	1	3	6	7	4	1			22
7	0	0	1	3	6	7	4	1		22
8	0	1	4	11	19	21	15	6	1	78

We illustrate the last for the case $k = 4, c = 5$. Each subsequence counted by $y_{8,6}$ is of just one of the shapes $*****8$, $****68$, $***678$, or $****78$, where $*$ represents a member other than 6, 7, or 8. The first three of these are formed by appending 8 to a subsequence of length 5, counted by $z_{7,5}$, and the last by appending 78 to a subsequence of length 4, counted by $x_{6,4}$ (none of which end in 6; note the correspondence between the subsequences counted by $x_{6,4}$ and those counted by $z_{5,4}$):

$$z_{2k} = 2z_{2k-1} + z_{2k-3}, \quad z_{2k+1} = 3z_{2k-1} + 2z_{2k-3}.$$

This last recurrence, which again holds for $k \geq 0$ with the aforementioned conventions, can be solved in the classical manner to show that

$$z_{2k+1} = \left(\frac{17+7\sqrt{17}}{34}\right)\left(\frac{3+\sqrt{17}}{2}\right)^k + \left(\frac{17-7\sqrt{17}}{34}\right)\left(\frac{3-\sqrt{17}}{2}\right)^k;$$

the previous formula then gives

$$z_{2k} = \left(\frac{17+3\sqrt{17}}{34}\right)\left(\frac{3+\sqrt{17}}{2}\right)^k + \left(\frac{17-3\sqrt{17}}{34}\right)\left(\frac{3-\sqrt{17}}{2}\right)^k.$$

Since the second term in these formulas tends rapidly to zero, we find that

$$z_n \text{ is the nearest integer to } c\zeta^n,$$

where

$$\zeta = \frac{1}{2}(3+\sqrt{17}) = 3.5615528128088302749107049 27$$

and

$$\begin{cases} c = \dfrac{1}{34}(17+3\sqrt{17}) = 0.8638034375544994602783596931 & \text{if } n \text{ is even,} \\ c = \dfrac{1}{34}(17+7\sqrt{17}) = 1.348874687627165407316172617 & \text{if } n \text{ is odd.} \end{cases}$$

We searched in our preview copy of [3] without any success, and were surprised that there seemed to be no earlier occurrences of members of our arrays. A similar problem with a similar but not very closely related answer is discussed in [1]. We then tried the main sequence $\{z_n\}$ on Superseeker [2], which produced the generating function

$$\sum_{j=0}^{\infty} z_j t^j = (1+t+2t^3)(1-(3t^2+2t^4))^{-1}$$

which should, perhaps, be thought of as the sum of two generating functions, one for odd-ranking terms, the other for even.

As the sequence does not seem to have been calculated earlier, we give a fair number of terms in the table below.

TABLE 4

n	z_n	n	z_n	n	z_n
1	1	17	34921	33	904069513
2	3	18	79647	34	2061980415
3	5	19	124373	35	3219891317
4	11	20	283667	36	7343852147
5	17	21	442961	37	11467812977
6	39	22	1010295	38	26155517271
7	61	23	1577629	39	40843221565
8	139	24	3598219	40	93154256107
9	217	25	5618809	41	145465290649
10	495	26	12815247	42	331773802863
11	773	27	20011685	43	518082315077
12	1763	28	45642179	44	1181629920803
13	2753	29	71272673	45	1845177526529
14	6279	30	162557031	46	4208437368135
15	9805	31	253841389	47	6571697209741
16	22363	32	578955451		

As may be expected from sequences defined from recurrence relations, there are congruence and divisibility properties. The terms of odd rank are alternately congruent to 1 and 5 modulo 8, and those of even rank after the second are congruent to 3 and 7 modulo 8 alternately. Every fourth term, starting with z_2 is divisible by 3, every third of those (e.g., z_{10}) is divisible by 9, every third of those (e.g., z_{34}) is divisible by 27, and so on. The terms that are divisible by 5 are every twelfth, starting with z_3 among the odd ranks and with z_{10} among those with even rank. Every sixteenth term is divisible by 7, starting with z_9 and z_{14}. Every sixth term is divisible by 11, starting with z_4; but no odd-ranking terms are. Seventeen is special to this sequence and divides every thirty-fourth term starting with $z_5 = 17$ itself and with z_{32} among the even-ranking terms.

Note that if you use the recurrences to calculate earlier terms in the sequence, $z_{-1} = 1$ and $z_{-3} = -1$, as we have already assumed, $z_{-2} = 0$ (and so is divisible in particular by 17, 11, 7, 5, and any power of 3), $z_{-4} = \frac{1}{2}$, $z_{-5} = 2$, $z_{-6} = -\frac{3}{4}$, $z_{-7} = -\frac{7}{2}$, $z_{-8} = \frac{11}{8}$, $z_{-9} = \frac{25}{4}$, ..., and there are p-adic interpretations of the divisibility properties. For example, z_{-9} is divisible by 25, and we leave it to the reader to confirm that $25|z_{51}$ and that $5^4|z_{171}$.

Why shouldn't the even numbers get equal time? If we denote by w_n the number of subsequences whose **even** members all have at least one **odd** neighbor, then for even $n = 2k$ there is the obvious symmetry $w_{2k} = z_{2k}$. The values of w_n for odd rank are the averages of the even-ranking neighbors: $2w_{2k+1} = w_{2k} + w_{2k+2}$, whereas for the $\{z_n\}$ sequence, the roles are reversed: $2z_{2k} = z_{2k-1} + z_{2k+1}$. Both sequences satisfy the recurrence $u_n = 3u_{n-2} + 2u_{n-4}$, while the generating function for $\{w_n\}$ has numerator $1 + 2t + t^3$ in place of $1 + t + 2t^3$.

TABLE 5. Some Odd-Ranking Members of the $\{w_n\}$ Sequence

n	w_n	n	w_n	n	w_n
−7	5/16	7	89	21	646981
−5	−1/8	9	317	23	2304257
−3	1/4	11	1129	25	8206733
−1	1/2	13	4021	27	29228713
1	2	15	14321	29	104099605
3	7	17	51005	31	370756241
5	25	19	181657	33	1320467933

The special role of 17 is illustrated by the form of w_{17} (i.e., $51005 = 5 \cdot 101^2$) and in the formula

$$w_{2k+1} = \left(\frac{17+4\sqrt{17}}{17}\right)\left(\frac{3+\sqrt{17}}{2}\right)^k + \left(\frac{17-4\sqrt{17}}{17}\right)\left(\frac{3-\sqrt{17}}{2}\right)^k.$$

More investigative readers will discover the many corresponding congruence and divisibility properties.

REFERENCES

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2. N. J. A. Sloane, S. Plouffe, & B. Salvy. The Superseeker program, email address: superseeker@research.att.com.
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