COUSINS OF SMITH NUMBERS: MONICA AND SUZANNE SETS

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(*Submitted February 1994*)

1. INTRODUCTION

For any $x \in \mathbb{N}$, x may be expressed as the sum $a_0 + a_1(10^1) + \cdots + a_d(10^d)$, where each $a_i \in \{0, 1, 2, \ldots, 9\}$.

Suppose that $x \in \mathbb{N}$ is a composite number. Then $x = p_1 p_2 \ldots p_m$, where each p_k is a prime number. We can then formally define two functions $S(x)$ and $S_p(x)$ as

$$S(x) = \sum_{j=0}^{d} a_j \quad \text{and} \quad S_p(x) = \sum_{i=1}^{m} S(p_i).$$

That is, $S(x)$ is the digital sum of x and $S_p(x)$ is the digital sum of the prime factors of x. Wilansky [2] defines a Smith number as a composite integer where $S(x) = S_p(x)$. This paper deals with two kinds of sets related to Smith numbers. These sets are called Monica sets and Suzanne sets.

Definition 1.1: The n^{th} Monica set $\mathbf{M}_n$ consists of those composite numbers x for which $n | S(x) - S_p(x)$ [we write $S(x) \equiv_n S_p(x)$].

Definition 2.1: The n^{th} Suzanne set $\mathbf{S}_n$ consists of those composite numbers x for which $n|S(x)$ and $n|S_p(x)$.

It should be noted that because I developed this concept from Smith numbers, I consider it to be akin to Smith's. Therefore, I have named these sets after my cousins, Monica and Suzanne Hammer.

2. ON THE POPULATION OF MONICA AND SUZANNE SETS

The following theorems give indications of what sort of integers belong to Monica and Suzanne sets.

Theorem 2.1: If x is a Smith number, then $x \in \mathbf{M}_n$, $\forall n \in \mathbb{N}$.

Theorem 2.2: $x \in \mathbf{S}_n \Rightarrow x \in \mathbf{M}_n$.

Note that the converse of Theorem 2.2 is *not* true; for example, $10 = 5 \times 2$, thus $S(10) = 1$ and $S_p(10) = 7$. $10 \in \mathbf{M}_6$ since $6|1-7$, but $10 \notin \mathbf{S}_6$ since $6 \nmid 1$.

Theorem 2.3: For any integer $\mathbf{k} > 1$, if x is a $\mathbf{k}$-Smith, then $x \in \mathbf{M}_{k-1}$.

Proof: McDaniel [1] defines a $\mathbf{k}$-Smith as a composite number x such that $\mathbf{k}S(x) = S_p(x)$. Thus, $S(x) - S_p(x)$ is divisible by $\mathbf{k} - 1$. Therefore, $x \in \mathbf{M}_{k-1}$. □

3. RELATIONS BETWEEN SETS OF MONICAS AND SETS OF SUZANNES

There are some rather simple properties of Monica and Suzanne sets that may be useful in later studies.

Theorem 3:

(a) If $p, q \in \mathbb{N}$ and $p|q$, then $x \in \mathbf{M}_q$ implies $x \in \mathbf{M}_p$;

(b) If $p, q \in \mathbb{N}$ and $p|q$, then $x \in \mathbf{S}_q$ implies $x \in \mathbf{S}_p$;

(c) If $p, q \in \mathbb{N}$ and $p|q$ are relatively prime, then $x \in \mathbf{M}_p$ and $x \in \mathbf{M}_q$ implies $x \in \mathbf{M}_{pq}$;

(d) If $p, q \in \mathbb{N}$ and $p|q$ are relatively prime, then $x \in \mathbf{S}_p$ and $x \in \mathbf{S}_q$ implies $x \in \mathbf{S}_{pq}$.

4. INFINITE ELEMENTS IN EACH MONICA AND SUZANNE SET

The most interesting property of Monica and Suzanne sets is that every Monica set and every Suzanne set has an infinite number of elements. McDaniel [1] proves that there is an infinite number of Smiths; this implies, by Theorem 2.1, that every Monica set has an infinite number of elements.. The proof that there is an infinite number of elements in each Suzanne set is more complicated.

Theorem 4: All Suzanne sets have infinitely many elements.

Proof: Consider $\mathbf{S}_1$. For any composite number x, $1|S(x)$ and $1|S_p(x)$.

For $\mathbf{S}_n$, where $n > 1$, we need to construct a candidate integer r such that $S(r) = n$. Let r be an n-digit *Repunit*, that is, a string of n ones (see [3]). Let $z = \alpha r$, where α is determined by the following table:

$$
\begin{array}{lll}
S_p(r) \equiv_7 0 & \text{then} \quad \alpha = 1 & \text{since } S_p(1r) = S_p(r) \\
S_p(r) \equiv_7 1 & \text{then} \quad \alpha = 9 & \text{since } S_p(9) = 6 \\
S_p(r) \equiv_7 2 & \text{then} \quad \alpha = 5 & \text{since } S_p(5) = 5 \\
S_p(r) \equiv_7 3 & \text{then} \quad \alpha = 4 & \text{since } S_p(4) = 4 \\
S_p(r) \equiv_7 4 & \text{then} \quad \alpha = 3 & \text{since } S_p(3) = 3 \\
S_p(r) \equiv_7 5 & \text{then} \quad \alpha = 2 & \text{since } S_p(2) = 2 \\
S_p(r) \equiv_7 6 & \text{then} \quad \alpha = 15 & \text{since } S_p(15) = 8 \equiv_7 1
\end{array}
$$

From the table it should be obvious that $7|S_p(r) + S_p(\alpha)$, and thus $7|S_p(z)$. Note that $S(z) = S(r)S(\alpha)$ because of our choice of r, so $n|S(z)$.

Let m be an integer such that $n|(S_p(z) + 7m)$ and let $y = z * 10^m$. Clearly, $S_p(y) = S_p(z) + S_p(10^m)$ and $S_p(10^m) = 7m$; thus, $n|S_p(y) = S_p(z) + S_p(10^m)$.

Note that $S(y) = S(z)$, so $n|S(y)$; thus, $\alpha r * 10^m = y \in \mathbf{S}_n$ for all m such that $n|S_p(\alpha r) + 7m$. Clearly, there are infinitely many choices for m. □

ACKNOWLEDGMENTS

I would like to thank my brother Matthew for his (nontechnical) assistance, Dr. Samuel Wagstaff at the Department of Computer Science at Purdue University for his most helpful suggestions, and *Sepultura* for the music video I watched when I came up with the linchpin of the proof for Theorem 4.

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AMS Classification Numbers: 11A63, 11A07