CONSTRUCTION OF SMALL CONSECUTIVE NIVEN NUMBERS

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1. DEFINITIONS AND NOTATION

A Niven number is a number divisible by its digital sum. In [1] it is shown there can exist at most twenty consecutive Niven numbers; moreover, an infinite family of such is constructed where the first example requires over 4 billion digits. Here we get a lower bound on the number of digits in each of twenty consecutive Niven numbers and construct an example with nearly this few digits.

We start by recalling

Definition: A positive integer is called a *Niven number* if it is divisible by its digital sum.

Example: The number 12 is a Niven number since $(1+2)|12$. The number 11 is not a Niven number since $(1+1)\nmid 11$.

For $n \in \mathbb{Z}$, let $s(n)$ denote the digital sum of n. Examples of 20 consecutive Niven numbers have a large number of digits. To represent these numbers nicely, we concatenate digits or blocks of digits. If a and b are blocks of digits, let ab be the concatenation of these blocks. To denote multiplication, we use $a * b$. Finally, a_k denotes the concatenation of k copies of a.

2. PRELIMINARY LEMMA

In [1], Cooper and Kennedy show that any sequence of twenty consecutive Niven numbers begins with a number congruent to 90 modulo 100. We push this idea further.

Definition: Let $n \in \mathbb{Z}^+$. A positive integer is a 10^n*-mark* if it is congruent to $9_{n-1}0$ modulo 10^n but not congruent to 9_n0 modulo 10^{n+1}.

What Cooper and Kennedy actually show is that a sequence of twenty consecutive Niven numbers must begin with a 10^{2*n_1}-mark for some $n_1 \in \mathbb{Z}^+$.

Lemma 1: A sequence of twenty consecutive Niven numbers begins with a 10^{280*n_2}-mark for some $n_2 \in \mathbb{Z}^+$.

Proof: Since twenty consecutive Niven numbers begin with a 10^{2*n_1}-mark for some $n_1 \in \mathbb{Z}^+$, the digital sums of the first ten Niven numbers in our sequence are consecutive, as are the digital sums of the second ten Niven numbers in our sequence. Therefore, there is at least one number, say m, in the first ten Niven numbers whose digital sum is divisible by 2^3. By the definition of a Niven number, this means m is divisible by 2^3. Similarly, there exists such a number, say m', among the second ten. We can take m and m' to differ by exactly 8. Then

$$s(m) - s(m') = 9 * 2 * n_1 - 8$$

as $2*n_1$ nines are converted to zeros upon crossing from the first to the second ten numbers if we start at a 10^{2*n_1}-mark. Since $9*2*n_1-8=s(m)-s(m')\equiv 0 \bmod 8$, we see that $2*n_1$ is a multiple of 8.

Using the same method, we see $2*n_1$ must also be a multiple of 5 and of 7. Taken together, this means that the first of our twenty consecutive Niven numbers is a 10^{280*n_2}-mark with $n_2=\frac{2*n_1}{280}\in\mathbb{Z}^+$. □

3. CONGRUENCE RESTRICTIONS OF THE DIGITAL SUM

In this section we develop congruence restrictions on the digital sum of our first Niven number. To do this, we assume β is the first of twenty consecutive Niven numbers, that it is a 10^{280*n_2}-mark for fixed $n_2\in\mathbb{Z}^+$, and that $s(\beta)=\alpha$. Then

$$(\alpha+i)|(\beta+i) \quad \text{for } i=0,1,\ldots,9,$$

$$(\alpha+j-2520*n_2)|(\beta+j) \quad \text{for} j=10,11,\ldots,19.$$

Let

$$\gamma=\operatorname{lcm}(\alpha,\alpha+1,\ldots,\alpha+9)$$

and

$$\gamma'=\operatorname{lcm}(\alpha+10-2520*n_2,\alpha+11-2520*n_2,\ldots,\alpha+19-2520*n_2).$$

This gives us

$$\beta\equiv\alpha \bmod \gamma$$

and

$$\beta\equiv\alpha-2520*n_2 \bmod \gamma'.$$

By the Chinese remainder theorem, these are consistent if and only if

$$\gcd(\gamma,\gamma')|2520*n_2. \tag{1}$$

This condition reduces to congruence conditions on the prime divisors of $A=\{2520*n_2-19, 2520*n_2-18,\ldots,2520*n_2-1\}$, the set of possible differences between $\{\alpha,\alpha+1,\ldots,\alpha+9\}$ and $\{\alpha+10-2520*n_2,\ldots,\alpha+19-2520*n_2\}$. Let P be the set of prime divisors of the numbers in A. For $p\in P$, let $v(p)$ be such that $p^{v(p)}|2520*n_2$ but $p^{v(p)+1}\nmid 2520*n_2$. Then condition (1) implies $p^{v(p)+1}\nmid\gcd(\gamma,\gamma')$, so $p^{v(p)+1}\nmid\gamma$ or $p^{v(p)+1}\nmid\gamma'$, which can be restated as

$$\begin{gathered}\alpha\equiv 1,2,\ldots,p^{v(p)+1}-10 \bmod p^{v(p)+1} \quad\text{or}\\ \alpha+10-2520*n_2\equiv 1,2,\ldots,p^{v(p)+1}-10 \bmod p^{v(p)+1}.\end{gathered}\tag{2}$$

Conversely, condition (2) assures $p^{v(p)+1}\nmid\gcd(\gamma,\gamma')$ and $p|\gcd(\gamma,\gamma')$ implies $p\in P$, so we get $\gcd(\gamma,\gamma')|2520*n_2$. Further, α satisfies some additional congruences modulo powers of 2 and 5. Since β is a 10^{280*n_2}-mark, $\beta\equiv 990 \bmod 1000$. Then $\beta\equiv 6 \bmod 8$, so

$$\alpha\equiv 6 \bmod 8. \tag{3}$$

Similarly $\beta\equiv 0 \bmod 5$ means

$$\alpha\equiv 0 \bmod 5. \tag{4}$$

These lead to

Lemma 2: A sequence of twenty consecutive Niven numbers must begin at a 10^{560*n_3}-mark for some $n_3 \in \mathbb{Z}^+$.

Proof: Suppose not, i.e., suppose we have twenty consecutive Niven numbers beginning at a 10^{280*n_2}-mark with n_2 odd. Then by (2) we have

$$\alpha \equiv 1, 2, \dots, 6 \bmod 16 \quad \text{or}$$
$$\alpha + 10 - 2520 * n_2 \equiv 1, 2, \dots, 6 \bmod 16,$$

so $\alpha \equiv 0, 1, \dots, 6, 15 \bmod 16$. By (3) this means $\alpha \equiv 6 \bmod 16$. Further, $\beta \equiv 9990 \bmod 10000$, so $\beta \equiv 6 \bmod 16$. Then $\alpha + 18 - 2520 * n_2 \equiv 0 \bmod 16$, so $16 | (\beta + 18)$, which is a contradiction. □

As a result of Lemma 2, we see that $2520 * n_2 \equiv 0 \bmod 16$, so the digital sums of $\beta, \beta+1, \dots, \beta+19$ are consecutive modulo 16. Since $\beta \equiv 6 \bmod 16$, we get

$$\alpha \equiv 6 \bmod 16. \tag{5}$$

4. OUTLINE OF METHOD

Assume $n_2 \in 2\mathbb{Z}^+$ fixed and α satisfies (2), (4), and (5). We outline how to construct the first of twenty consecutive Niven numbers β so that $s(\beta) = \alpha$. By our choice of α, we can find a solution to

$$x \equiv \alpha - 2520 * n_2 \bmod \gamma' \tag{6}$$

and

$$x \equiv \alpha \bmod \gamma. \tag{7}$$

In fact, we can find infinitely many solutions differing from each other by multiples of $\delta = \operatorname{lcm}(\gamma, \gamma')$. Let b be the least positive solution. We modify b by adding multiples of δ so that the resulting number, b', still satisfies (6) and (7) and is a 10^{280*n_2}-mark. Finally, we may be able to modify b' by adding multiples of δ so that the resulting number, β, still satisfies (6) and (7), is still a 10^{280*n_2}-mark, and has a digital sum α. Such a β is the first of twenty consecutive Niven numbers.

5. AN EXAMPLE OF A SEQUENCE OF SMALL NIVEN NUMBERS

We construct a 10^{280*4}-mark. This means $n_2 = 4$. We can solve the congruences (2), (4), and (5) for α modulo $p \in P$ to get $\alpha = 15830$. This leads to

$$\begin{aligned}\delta &= \operatorname{lcm}(\alpha, \alpha+1, \dots, \alpha+19-10080)\\ &= 30488306558784378902267998668166603\\ &\quad\ 21626948228266570463950023607020 80.\end{aligned}$$

Solving for x in (6) and (7), we get

$$\begin{aligned} b = x &= 36346620873326536780272919778661 48\\ &\quad\ 019043614233737117568189046296950.\end{aligned}$$

Adding a suitable multiple of δ (to get a 10^{1120}-mark), we get

$$\begin{aligned} b' &= 21222185596541538670917359810786534\\ &\quad\ 262151758227361053517782513105 9_{1119}0.\end{aligned}$$

Continue to add multiples of δ so as not to disturb the terminal 1121 digits of b'. In short, add multiples of $5^6 * 10^{1114} * \delta$. Doing this, we get

$$\begin{aligned}\beta = {} & 4981497945879639583073518757993538 2447\\ & 7644885805506055872527914072 9_{601} 5 9_{1119} 0.\end{aligned}$$

It is easy to check that this is a number with 1788 digits and digital sum 15830 and that β is the first of twenty Niven numbers.

6. LOWER BOUNDS ON THE NUMBER OF DIGITS

Theorem 1: The smallest sequence of twenty consecutive Niven numbers begins with a 10^{1120}-mark of digital sum 15830.

Proof: Let β be the first of twenty consecutive Niven numbers and let $\alpha = s(\beta)$. Suppose β has fewer than 1789 digits (i.e., no more than in our example in the previous section). Since β is a 10^{280*n_2}-mark with n_2 even, $n_2 = 2, 4$, or 6. A computer search shows there is no α less than $9 * 1789$ satisfying (2), (4), and (5) with $n_2 = 2, 4$, or 6 other than $\alpha = 15830$ for $n_2 = 4$. □

In the last section we saw an example with 1788 digits. This need not be the smallest, but it is close to the smallest.

Theorem 2: The smallest sequence of twenty consecutive Niven numbers begins with a number having at least 1760 digits.

Proof: By Theorem 1, we know the digital sum of the first number is 15830. Given that the terminal digit is a zero, there are at least 1+15830/9 digits. □

REFERENCE

1. C. Cooper & R. Kennedy. "On Consecutive Niven Numbers." *The Fibonacci Quarterly* **31.2** (1993):146-51.

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