P-LATIN MATRICES AND PASCAL'S TRIANGLE MODULO A PRIME

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(*Submitted December 1994*)

INTRODUCTION

One of the more effective methods of counting residues modulo a prime in the rows of Pascal's triangle is a reduction of this problem to that of solving of certain systems of recurrence equations. This way was successfully employed by B. A. Bondarenko [1] in the investigation of this problem for various values of p and (only) for certain rows of Pascal's triangle. However, some characteristic properties of the matrices of these recurrent systems were noticed which led to the idea of p-latin matrices. This idea was formulated in more detail in [2], which also uses p-latin matrices in the investigation of other arithmetic triangles.

In this paper we consider a new application of the properties of p-latin matrices to the investigation of Pascal's triangle modulo a prime. Using a representation of the p-latin matrices in a convenient basis, we obtain the distribution of Pascal's triangle elements modulo a prime for an arbitrary row.

p-LATIN MATRICES

We note the definition of a p-latin matrix as given in [1] and [2]. A square matrix of order n is called a "latin square of order n" [3] if its elements take on n values in such a way that each value occurs only once in each column and row. A latin square of order n is called a "p-latin square of order n" if no diagonals except the main and secondary ones (the element indices are i and $n-i+1$ for $1 \le i \le n$) have equal elements. A p-latin square of order n is said to be a "normalized p-latin square of order n" if its first row has the form $(1, 2, \ldots, n)$, and the main diagonal has the form $(1, \ldots, 1)$.

We will construct such a matrix for any prime p.

Let us introduce the matrix $P = (j/i)_{i,j=\overline{1,p-1}}$ of order $p-1$ whose elements are to be understood as elements from the field $\mathbb{Z}_p$. (Here and later we use the notation $i, j = \overline{1, p-1}$ to mean $1 \le i \le p-1, 1 \le j \le p-1$.)

Example 1: For $p = 7$, the matrix P has the form:

$$P = \begin{pmatrix} 1/1 & 2/1 & 3/1 & 4/1 & 5/1 & 6/1 \\ 1/2 & 2/2 & 3/2 & 4/2 & 5/2 & 6/2 \\ 1/3 & 2/3 & 3/3 & 4/3 & 5/3 & 6/3 \\ 1/4 & 2/4 & 3/4 & 4/4 & 5/4 & 6/4 \\ 1/5 & 2/5 & 3/5 & 4/5 & 5/5 & 6/5 \\ 1/6 & 2/6 & 3/6 & 4/6 & 5/6 & 6/6 \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 3 & 1 & 6 & 4 & 2 \\ 2 & 4 & 6 & 1 & 3 & 5 \\ 3 & 6 & 2 & 5 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Theorem 1: If p is a prime number, then the matrix P is a normalized p-latin square.

Proof: It is obvious that elements of P occurring in the same row or column are distinct and belong to the multiplicative group of the field $\mathbb{Z}_p$. Thus the matrix P is a latin square.

Let j/i be one element of some diagonal that is parallel to the main diagonal. Then any other element of this diagonal has the form $(j+s)/(i+s)$. Assume that these elements are equal; then $is = js$ and therefore $i = j$, so in this case the element j/i has to occur on the main diagonal. There is an analogous situation with diagonals parallel to the secondary one. Hence P is a p-latin square. Since the first row of P has the form $1, 2, \ldots, p-1$ and on the main diagonal there are only 1's, P is a normalized p-latin square.

Let us define the set of square matrices of order $p-1$ (called in [2] "normalized p-latin matrices"):

$$\mathbb{N}_p = \left\{ \left(c_{p_{i,j}}\right)_{i,\, j=\overline{1,\, p-1}} \middle| c_1, \ldots, c_{p-1} \in \mathbb{C}, (p_{i,j}) = P \right\},$$

where $\mathbb{C}$ denotes the complex numbers.

Example 2: If $p = 7$, then, according to Example 1, the matrix

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_4 & c_1 & c_5 & c_2 & c_6 & c_3 \\ c_5 & c_3 & c_1 & c_6 & c_4 & c_2 \\ c_2 & c_4 & c_6 & c_1 & c_3 & c_5 \\ c_3 & c_6 & c_2 & c_5 & c_1 & c_4 \\ c_6 & c_5 & c_4 & c_3 & c_2 & c_1 \end{pmatrix}$$

belongs to $\mathbb{N}_7$.

Though the idea of this set of matrices was contained in [1] and [2], their existence for any p was not made explicit.

Corollary 1: If $C, B \in \mathbb{N}_p$, then $CB \in \mathbb{N}_p$ and $CB = BC$.

Proof: In fact, if $C = (c_{i,j})$ and $B = (b_{i,j})$, then the equality

$$CB = \left(\sum_{k=1}^{p-1} c_{k/i} b_{j/k} \right)_{i,\, j=\overline{1,\, p-1}} = \left(\sum_{s=1}^{p-1} c_s b_{j/(is)} \right)_{i,\, j=\overline{1,\, p-1}},$$

where all indices are in $\mathbb{Z}_p$, holds. Therefore, if we denote by a_k the sum $\sum_{s=1}^{p-1} c_s b_{k/s}$, then we will have $CB = (a_{j/i})_{i,\, j=\overline{1,\, p-1}}$; hence $CB \in \mathbb{N}_p$. Moreover, in the same way, we can establish

$$BC = \left(\sum_{s=1}^{p-1} b_s c_{js/i} \right)_{i,\, j=\overline{1,\, p-1}} = (a_{j/i})_{i,\, j=\overline{1,\, p-1}},$$

with the aid of the equality

$$a_k = \sum_{s=1}^{p-1} b_s c_{k/s}.$$

Hence $CB = BC$, which was to be proved.

We develop the properties of these matrices from $\mathbb{N}_p$ in what follows.

Let us denote by $\Delta^{(1)}$ the Pascal triangle modulo a prime p and let $C(n,m)$ be an arbitrary element. Let us also denote by $\Delta_s^{(1)}$ the triangle containing only the first s rows of $\Delta^{(1)}$. Now consider the triangle $\Delta^{(k)} \equiv k\Delta^{(1)}$, whose elements $C_k(n,m)$ are defined by the expression $C_k(n,m) = kC(n,m) \pmod p$ and denote by $\Delta_s^{(k)}$ the triangle containing only the first s rows of $\Delta^{(k)}$. It is clear that $\Delta_s^{(k)} = k\Delta_s^{(1)}$.

Definition: The triangle with sm rows arising from $\Delta_s^{(k)}$ by replacing its elements $C(n,\ell)$ by the triangles $\Delta_m^{(C(n,\ell))}$ and filling in free places by 0 is denoted by $\Delta_s^{(k)} * \Delta_m$.

Example 3: For $p=5$, the triangles $\Delta_4^{(1)}$ and $\Delta_3^{(k)}$ have the form

$$\Delta_4^{(1)} = \begin{matrix} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \end{matrix}, \quad \Delta_3^{(1)} = \begin{matrix} 1 \\ 1\ 1 \\ 1\ 2\ 1 \end{matrix}, \quad \Delta_3^{(2)} = \begin{matrix} 2 \\ 2\ 2 \\ 2\ 4\ 2 \end{matrix}, \quad \Delta_3^{(3)} = \begin{matrix} 3 \\ 3\ 3 \\ 3\ 1\ 3 \end{matrix},$$

and therefore we obtain:

$$\Delta_4^{(1)} * \Delta_3 \equiv \begin{matrix} \Delta_3^{(1)} \\ \Delta_3^{(1)}\ \Delta_3^{(1)} \\ \Delta_3^{(1)}\ \Delta_3^{(2)}\ \Delta_3^{(1)} \\ \Delta_3^{(1)}\ \Delta_3^{(3)}\ \Delta_3^{(3)}\ \Delta_3^{(1)} \end{matrix} = \begin{matrix} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1 \\ 1\ 2\ 1\ 1\ 2\ 1 \\ 1\ 0\ 0\ 2\ 0\ 0\ 1 \\ 1\ 1\ 0\ 2\ 2\ 0\ 1\ 1 \\ 1\ 2\ 1\ 2\ 4\ 2\ 1\ 2\ 1 \\ 1\ 0\ 0\ 3\ 0\ 0\ 3\ 0\ 0\ 1 \\ 1\ 1\ 0\ 3\ 3\ 0\ 3\ 3\ 0\ 1\ 1 \\ 1\ 2\ 1\ 3\ 1\ 3\ 3\ 1\ 3\ 1\ 2\ 1 \end{matrix}$$

This leads to the principal fractal property of Pascal's triangle.

Theorem 2: For any $n, m \in \mathbb{N}$ and each $k = \overline{1, p-1}$, the equality $\Delta_m^{(k)} * \Delta_{p^n} = \Delta_{mp^n}^{(k)}$ holds.

The proof of this theorem is lengthy but not difficult and is given in [4].

This result allows us to reduce an investigation of $\Delta^{(1)}$ to the investigation of $\Delta_p^{(k)}$ for $k = \overline{1, p-1}$. The details will be given in Theorem 3.

Let B_k, $1 \le k \le p-1$, be the matrix of order $p-1$, any element $b_{i,j}$ of which is the number of elements equal to j in the k^{th} row of the triangle $\Delta_p^{(i)}$. Denote by $g_s^{(k)}(n,p)$ the number of elements equal to s modulo p in the n^{th} row of the triangle $\Delta^{(k)}$.

Theorem 3: If $n = (a_r, \ldots, a_0)_p$ is the p-ary representation of n, then

$$g_s^{(k)}(n,p) = (B_{a_r} \ldots B_{a_0})_{k,s}. \tag{1}$$

Proof: Using Theorem 2, we can write the equality

$$\Delta^{(1)}_{p^{r+1}} = \Delta^{(1)}_{p} * \Delta_{p^r},$$

which means that the n^{th} row of $\Delta^{(1)}_{p^{r+1}}$ is found in the a_r^{th} row of $\Delta^{(1)}_{p}$, which consists of the triangles $\Delta^{(k)}_{p^r}, 1 \le k \le p-1$ (see Example 3). If we set $n_{(k)} \equiv (a_{r-k}, \dots, a_0)$, then the following vector equality will hold:

$$(g_s^{(k)}(n, p))_{k=\overline{1, p-1}} = B_{a_r}(g_s^{(k)}(n_{(1)}, p))_{k=\overline{1, p-1}}.$$

Continuing this process, we can obtain

$$(g_s^{(k)}(n, p))_{k=\overline{1, p-1}} = B_{a_r} \dots B_{a_i}(g_s^{(k)}(n_{(r)}, p))_{k=\overline{1, p-1}}.$$

Since $n_{(r)} = a_0$ and $g_s^{(k)}(a_0, p) = (B_{a_0})_{s,k}$, we get (1). This completes the proof.

Using Theorem 3, we can reduce counting the $g_s^{(1)}(n, p)$, where $s = \overline{1, p-1}$, to finding a product of the matrices B_k.

Theorem 4: $B_k \in \mathbb{N}_p$.

Proof: Let $b_1^{(k)}, \dots, b_{p-1}^{(k)}$ be the elements of the first row of B_k. We will prove the equality

$$B_k = (b^{(k)}_{p_{i,j}})_{i,\, j=\overline{1, p-1}}. \tag{2}$$

We can define the addition of the triangles $\Delta^{(k)}_p$ as the same operation between corresponding elements of $\Delta^{(k)}_p$ in $\mathbb{Z}_p$. For example, the following equality

$$\sum_{k=1}^{s} \Delta^{(1)}_p = \Delta^{(s)}_p \tag{3}$$

holds. If we denote the elements of matrix B_k by $b_{i,j}^{(k)}$, then, using (3) and the definition of $b_{i,j}^{(k)}$, we can write $b_{1,j}^{(k)} = b_{s,js}^{(k)}$ for each $s = \overline{1, p-1}$. Thus, $b_{i,j}^{(k)} = b_{1,j/i}^{(k)}$, and hence (2) holds. The proof is complete.

Let n_i be the number of elements equal to i in the p-ary representation of n in the form $n = (a_r, \dots, a_0)_p$. By (1), using Corollary 1, we can find

$$g_s^{(k)}(n, p) = \left(\prod_{i=1}^{p-1} B_i^{n_i} \right)_{k,s}. \tag{4}$$

Here the matrix B_0 is absent because $B_0 = \operatorname{diag}(1, \dots, 1) \equiv E$. Now, to calculate the value of $g_s^{(k)}(n, p)$, we have to investigate the further properties of the matrices in $\mathbb{N}_p$.

PROPERTIES OF THE MATRICES FROM $\mathbb{N}_p$

It is true that $\mathbb{N}_p$ is just a subspace of the linear space of square matrices of order $p-1$. Moreover, we have

Corollary 2: $\operatorname{Dim}\mathbb{N}_p = p-1$ and

$$B \in \mathbb{N}_p \Rightarrow B = \sum_{k=1}^{p-1} b_k I_k, \tag{5}$$

where $I_k \in \mathbb{N}_p$ and $I_k = (\delta_{ki,j})_{i,j=\overline{1,p-1}}$.

Here $\delta_{i,j}$ is Kronecker's symbol and all indices are to be understood as elements from $\mathbb{Z}_p$.

Proof of this property can be obtained directly from the definition of $\mathbb{N}_p$.

Let us verify that the matrices I_k possess the property $I_k I_m = I_{km}$. In fact

$$I_k I_m = \left(\sum_{s=1}^{p-1} \delta_{ki,s}\, \delta_{ms,j} \right)_{i,j=\overline{1,p-1}},$$

and consequently the element of the matrix $I_k I_m$ with the indices i and j does not vanish if there exists an s so that $ki = s$ and $ms = j$. Hence $j = mki$, and therefore $I_k I_m = (\delta_{mki,j})_{i,j=\overline{1,p-1}} = I_{km}$.

Let ν be the root of the equation $x^{p-1} = 1$ in the field $\mathbb{Z}_p$, such that for each $k = \overline{1, p-2}$ the inequality $\nu^k \neq 1$ holds. For what follows, it will be convenient to introduce the matrices $J_k = (I_\nu)^k$. If we set $c_k = b_{\nu^k}$, then (5) can be written in the form

$$B = \sum_{k=1}^{p-1} c_k J_k. \tag{6}$$

Corollary 3: If μ is an eigenvalue of B, then there is a root of the equation $z^{p-1} = 1$ in $\mathbb{C}$, which we denote as λ, such that

$$\mu = \sum_{k=1}^{p-1} c_k \lambda^k. \tag{7}$$

Proof: Let a be some vector from $\mathbb{C}^{p-1}$ and

$$b = \sum_{k=1}^{p-1} \lambda^{-k} J_k a.$$

Then, employing the equality $J_s b = \lambda^s b$ and carrying this out for each $s = \overline{1, p-1}$, we can write

$$Bb = \sum_{k=1}^{p-1} c_k J_k b = \sum_{k=1}^{p-1} c_k \lambda^k b = \mu b,$$

i.e., μ is an eigenvalue of B. Now it remains to prove that formula (7) gives us all eigenvalues of B. We will complete this after Corollary 6.

As a consequence of Corollary 3, we note that the matrices I_k, and hence the matrices J_k, are nonsingular matrices, and $\forall k, \det I_k = \det J_k = 1$. Indeed, since all eigenvalues of J_k are the roots of the equation $x^{p-1} = 1$ (we denote them by λ_i), then we have

$$\det J_k = \prod_{i=1}^{p-1} \lambda_i^k = \mu^k,$$

where $\mu = \lambda_1 \ldots \lambda_{p-1}$. Using the equality $\sum_{k=1}^{p-1} k = 0 \pmod p$, we get $\mu = 1$, and hence $\det J_k = 1$.

As another interesting property of the matrices I_k we note that they are orthogonal matrices, namely, $I_k I_k^* = E$, where $(a_{i,j})^* = (\bar{a}_{j,i})$ and the bar denotes complex conjugation. This immediately follows from the equality

$$I_k^* = (\delta_{i,jk})_{i,\, j=\overline{1,\, p-1}} = I_{1/k}.$$

Obviously, the matrices J_k possess the same property, but by the equality $J_k J_s = J_{k+s}$ we have

$$J_k^* = J_k^{-1} = J_{p-k-1} \tag{8}$$

for each $k = \overline{1, p-2}$. Since $J_{p-1} = E$, we have $J_{p-1}^* = J_{p-1}$.

Corollary 4: Let B be in $\mathbb{N}_p$ and be written in the form (6), then

$$B^* = \sum_{k=1}^{p-2} \bar{c}_{p-k-1} J_k + \bar{c}_{p-1} J_{p-1}.$$

Proof: In fact, using (8), we immediately obtain

$$B^* = \sum_{k=1}^{p-2} \bar{c}_k J_k^* + \bar{c}_{p-1} J_{p-1}^* = \sum_{k=1}^{p-2} \bar{c}_k J_{p-k-1} + \bar{c}_{p-1} J_{p-1};$$

hence Corollary 4 is true.

Let us introduce the matrices S_i for $i = \overline{1, p-1}$ in the form

$$S_i = \frac{1}{(p-1)} \sum_{k=1}^{p-1} \lambda_i^{-k} J_k. \tag{9}$$

Here, as before, λ_i is one of the roots of the equation $x^{p-1} = 1$ in $\mathbb{C}$. It is clear that, for each $i = \overline{1, p-1}$, the matrices S_i belong to $\mathbb{N}_p$.

Let λ be a primitive root of the equation $x^{p-1} = 1$, i.e., for each $k = \overline{1, p-2}$, we have $\lambda^k \neq 1$. Therefore, in formula (9), we can assume that $\lambda_i = \lambda^i$.

Theorem 5: The following equalities,

$$S_i S_j = \delta_{i,j} S_i \tag{10}$$

are true for all $i, j = \overline{1, p-1}$.

Proof: Consider the left-hand side of (10). After some calculation, we get

$$S_i S_j = \frac{1}{(p-1)^2} \left[\sum_{\ell=0}^{p-2} \lambda_j^{-\ell} J_\ell \sum_{k=0}^{\ell} \lambda_{i-j}^{-k} + \sum_{\ell=p-1}^{2(p-2)} \lambda_j^{-\ell} J_\ell \sum_{k=\ell-p+2}^{p-2} \lambda_{i-j}^{-k} \right],$$

whence

$$S_i S_j = \frac{1}{(p-1)^2} \left[\sum_{\ell=0}^{p-2} \lambda_j^{-\ell} J_\ell \sum_{k=0}^{\ell} \lambda_{i-j}^{-k} + \sum_{\ell=0}^{p-3} \lambda_j^{-\ell} J_\ell \sum_{k=\ell+1}^{p-2} \lambda_{i-j}^{-k} \right];$$

hence

$$S_iS_j = \frac{1}{(p-1)^2}\sum_{\ell=0}^{p-2}\lambda_j^{-\ell}J_\ell\sum_{k=0}^{p-2}\lambda_{i-j}^{-k}.$$

Let us examine this equality. Employing the identity $\lambda_{i-j} = \lambda_i / \lambda_j$, where $\lambda_i \neq \lambda_j$ (for $i \neq j$), we obtain

$$\sum_{k=0}^{p-2}\lambda_{i-j}^{-k} = (\lambda_{i-j}^{1-p}-1)/(\lambda_{i-j}^{-1}-1) = 0.$$

Hence (10) holds for $i \neq j$. Further, at $i = j$, we have

$$\sum_{k=0}^{p-2}\lambda_{i-j}^{-k} = p-1; \tag{11}$$

consequently, $S_i^2 = S_i$, and the proof is complete.

The matrices S_i are Hermitian, i.e., they possess the property $S_i = S_i^*$. In fact, for $i = \overline{1, p-1}$, we have

$$S_i^* = \frac{1}{(p-1)}\sum_{k=1}^{p-1}\lambda_i^k J_k^* = \frac{1}{(p-1)}\sum_{k=1}^{p-1}\lambda_i^{k-p+1}J_{p-k-1} = S_i.$$

Let us denote the transposed matrix $A = (a_{i,j})_{i,j=\overline{1,p-1}}$ by $A' = (a_{j,i})_{i,j=\overline{1,p-1}}$. Then we have $S_i' = S_{p-i-1}$ for $i = \overline{1, p-2}$. This can be proved in the same way as the previous result, but we need to keep in mind that $J_k^* = J_k'$ and $\overline{\lambda}_i = \lambda_{p-i-1}$.

Theorem 6: The equalities

$$J_k = \sum_{i=1}^{p-1}\lambda_i^k S_i,\ k = \overline{1, p-1}, \tag{12}$$

which are converse to (9), are true.

Proof: Employing (9)-(11) and making some transformations, we get

$$\sum_{i=1}^{p-1}\lambda_i S_i = \sum_{k=1}^{p-1}\left[\frac{1}{(p-1)}\sum_{i=1}^{p-1}\lambda_i^{1-k}\right]J_k = \sum_{k=1}^{p-1}\delta_{k,1}J_k.$$

Therefore, (12) is true for $k = 1$. For the completion of the proof, it suffices to note that $J_k = J_1^k$ and to make use of (10).

Now we must note that the matrix S_{p-1} consists only of 1's in each place; hence $S_{p-1}' = S_{p-1}$. This is clear from the following equalities,

$$S_{p-1} = \sum_{k=1}^{p-1}J_k = \sum_{k=1}^{p-1}I_k = \left(\sum_{k=1}^{p-1}\delta_{ki,j}J_k\right)_{i,j=\overline{1,p-1}},$$

if we bear in mind that, for $i, j = \overline{1, p-1}$, $\sum_{k=1}^{p-1}\delta_{ki,j} = 1$.

Corollary 5: Let $B \in \mathbb{N}_p$, then

$$B = \sum_{i=1}^{p-1}\mu_i S_i, \quad \text{where } \mu_i \text{ are the eigenvalues of } B. \tag{13}$$

The proof of this Corollary can be obtained without difficulty from (6) by using Theorem 6 and equality (7).

Using the basis $S_1, \ldots, S_{p-1}$, we can easily find the product of matrices from $\mathbb{N}_p$. To illustrate this statement we prove

Theorem 7: Let $\mu_1^{(i)}, \ldots, \mu_{p-1}^{(i)}$ be the eigenvalues of the matrices B_i from Theorem 3. If we set

$$\sigma_j = \prod_{i=1}^{p-1} (\mu_j^{(i)})^{n_i}, \tag{14}$$

then the equality

$$g_s^{(k)}(n, p) = \left(\sum_{i=1}^{p-1} \sigma_i S_i \right)_{k,s} \tag{15}$$

is true.

Proof: It is readily seen that, making use of (13) and Theorem 5, we can obtain

$$B_i^{n_i} = \sum_{j=1}^{p-1} (\mu_j^{(i)})^{n_i} S_j.$$

Therefore, equality (4) transforms to (15), and the proof is complete.

Note that we can also write σ_j in the form $\sigma_j = \mu_j^{(a_r)} \ldots \mu_j^{(a_0)}$.

Corollary 6: Any eigenvector b_i of the matrix B corresponding to the eigenvalue μ_i can be written in the form

$$b_i = \sum_{(j)} S_j c_j, \tag{16}$$

where $c_j \in \mathbb{C}^{p-1}$ and the summation is taken over j satisfying the condition $\mu_j = \mu_i$.

Proof: Let b_i be the eigenvector of the matrix B corresponding to the eigenvalue μ_i. Operating on the equality $Bb_i = \mu_i b_i$ by the matrix S_s, using (13) and Theorem 5, we obtain $\mu_s S_s b_i = \mu_i S_s b_i$. If $\mu_i \neq \mu_s$ here, then $S_s b_i = 0$. Now, if we make use of the identity $E = S_1 + \cdots + S_{p-1}$, which easily follows from Corollary 5 for $B = E$, then we get $b_i = (\Sigma_{(j)} S_j) b_i$.

In addition, if $c \in \mathbb{C}^{p-1}$, then, using the equality $BS_i c = \mu_i S_i c$, we can say that the vectors of the form $S_i c$ are the eigenvectors corresponding to the eigenvalue μ_i. Thus (16) is true, and the proof is complete.

Conclusion of the Proof of Corollary 3: Let us take $c \in \mathbb{C}^{p-1}$ so that $\forall k, S_k c \neq 0$. This is possible, for example, with $c = (1, 0, \ldots, 0)$. We saw above that the vector $c_k = S_k c$ is the eigenvector of the matrix B corresponding to the eigenvalue μ_k determined from (7) at $\lambda = \lambda_k$. We claim that the vectors c_k $(k = \overline{1, p-1})$ are linearly independent. In fact, if there are $\delta_1, \ldots, \delta_{p-1} \in \mathbb{C}$ not all zero and such that $\delta_1 c_1 + \cdots + \delta_{p-1} c_{p-1} = 0$, then operating on this equality by S_k, we obtain $\delta_k c_k = 0$ or $\delta_k = 0$ for $k = \overline{1, p-1}$, which is a contradiction. Thus, the vectors c_k for $k = \overline{1, p-1}$ are the basis in $\mathbb{C}^{p-1}$, and so there are no other eigenvalues of B. Thus, the proof is complete.

***Corollary** 7:* If $\mu_i \neq 0$ for each $i = \overline{1, p-1}$, then the matrix B has an inverse defined by the equality

$$B^{-1} = \sum_{i=1}^{p-1} \mu_i^{-1} S_i .$$

To prove this statement, it is sufficient to use the identity $E = S_1 + \cdots + S_{p-1}$ again, and to employ Theorem 5.

Now we apply the properties obtained of the matrices from $\mathbb{N}_p$ to counting $g_s^{(k)}(n, p)$ for $p = 7$. It should be pointed out that in [5] this problem was considered for $p = 3$ and $p = 5$.

COUNTING $g_s^{(k)}(n, 7)$

To count the value of $g_s^{(k)}(n, p)$ we need, according to Theorem 7, to examine the triangles $\Delta_7^{(k)}$ for $k = \overline{1, 6}$. The triangle $\Delta_7^{(1)}$ has the form:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 3 & & 3 & & 5 & & 1 & \\
1 & & 6 & & 1 & & 6 & & 1 & & 6 & & 1
\end{array}$$

If we multiply each element of $\Delta_7^{(1)}$ by k in $\mathbb{Z}_p$, we will obtain the triangle $\Delta_7^{(k)}$. For example, $\Delta_7^{(3)}$ has the form:

$$\begin{array}{ccccccccccccc}
 & & & & & & 3 & & & & & & \\
 & & & & & 3 & & 3 & & & & & \\
 & & & & 3 & & 6 & & 3 & & & & \\
 & & & 3 & & 2 & & 2 & & 3 & & & \\
 & & 3 & & 5 & & 4 & & 5 & & 3 & & \\
 & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & \\
3 & & 4 & & 3 & & 4 & & 3 & & 4 & & 3
\end{array}$$

Now we need to find the matrices B_k for $k = \overline{1, 6}$. Let us take, for instance, the 4th rows of triangles $\Delta_7^{(k)}$, which give us the matrix B_4. The 4th row of triangle $\Delta_7^{(1)}$ has the form $(1, 4, 6, 4, 1)$. Since the numbers 1 and 4 occur twice and the number 6 occurs once there, the first row of B_4 has the form $(2, 0, 0, 2, 0, 1)$. If we want to count the third row of B_4 now, we must take the 4th row of triangle $\Delta_7^{(3)}$, which gives us what we desire, i.e., $(0, 0, 2, 1, 2, 0)$. Thus, we can count all the matrices B_k for $k = \overline{1, 6}$. To write our calculation, we make use of the matrices J_k $(k = \overline{1, 6})$. So let us find the matrix J_1. In our case, we have $\nu = 3$ because, for each $k = \overline{1, 5}$, the inequality $3^k \neq 1 \pmod 7$ is correct. Therefore,

$$J_1 = I_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now we can write

$$B_0 = J_6,\ \ B_1 = 2J_6,\ \ B_2 = J_2 + 2J_6,\ \ B_3 = 2J_1 + 2J_6,$$
$$B_4 = J_3 + 2J_4 + 2J_6,\ \ B_5 = 2J_1 + 2J_5 + 2J_6,\ \ B_6 = 3J_3 + 4J_6.$$

Let us assume that the number k is contained in the record of $(n)_7$ a total of n_k times. Using the notation of Theorem 7 and formulas (6) and (7), and keeping in mind that $\lambda_k = \exp(ik\pi/3)$ (here, $i^2 = -1$), we obtain, for each $k = \overline{1,6}$,

$$\mu_k^{(1)} = 2,\ \ \mu_k^{(2)} = \lambda_k^2 + 2,\ \ \mu_k^{(3)} = 2\lambda_k + 2,\ \ \mu_k^{(4)} = \lambda_k^3 + 2\lambda_k^4 + 2,$$
$$\mu_k^{(5)} = 2(\lambda_k + \lambda_k^5 + 1),\ \ \mu_k^{(6)} = 3\lambda_k^3 + 4.$$

Whence, by (14),

$$\begin{aligned} &\sigma_1 = 2^{n_1-n_2}(3+i\sqrt{3})^{n_2+n_3}(-i\sqrt{3})^{n_4}4^{n_5}, \\ &\sigma_2 = 2^{n_1-n_2}(3-i\sqrt{3})^{n_2}(1+i\sqrt{3})^{n_3}(2+i\sqrt{3})^{n_4}(2\lambda_2+2\lambda_4+2)^{n_5}7^{n_6}, \\ &\sigma_3 = (-1)^{n_5}2^{n_1+n_5}3^{n_2+n_4}(2\lambda_3+2)^{n_3},\ \ \sigma_6 = 2^{n_1}3^{n_2}4^{n_3}5^{n_4}6^{n_5}7^{n_6}, \\ &\sigma_4 = \overline{\sigma}_2,\ \ \sigma_5 = \overline{\sigma}_1, \end{aligned} \tag{17}$$

where the bar denotes the complex conjugate. To make use of (15), we need the matrices S_k $(k = \overline{1,6})$. According to (9), the matrices S_1 and S_2 have the form

$$S_1 = \frac{1}{6}\begin{pmatrix} 1 & \overline{\lambda}_2 & \overline{\lambda}_1 & \lambda_2 & \lambda_1 & -1 \\ \lambda_2 & 1 & \lambda_1 & \overline{\lambda}_2 & -1 & \overline{\lambda}_1 \\ \lambda_1 & \overline{\lambda}_1 & 1 & -1 & \lambda_2 & \overline{\lambda}_2 \\ \overline{\lambda}_2 & \lambda_2 & -1 & 1 & \overline{\lambda}_1 & \lambda_1 \\ \overline{\lambda}_1 & -1 & \overline{\lambda}_2 & \lambda_1 & 1 & \lambda_2 \\ -1 & \lambda_1 & \lambda_2 & \overline{\lambda}_1 & \overline{\lambda}_2 & 1 \end{pmatrix}, \quad S_2 = \frac{1}{6}\begin{pmatrix} 1 & \lambda_2 & \overline{\lambda}_2 & \overline{\lambda}_2 & \lambda_2 & 1 \\ \overline{\lambda}_2 & 1 & \lambda_2 & \lambda_2 & 1 & \overline{\lambda}_2 \\ \lambda_2 & \overline{\lambda}_2 & 1 & 1 & \overline{\lambda}_2 & \lambda_2 \\ \lambda_2 & \overline{\lambda}_2 & 1 & 1 & \overline{\lambda}_2 & \lambda_2 \\ \overline{\lambda}_2 & 1 & \lambda_2 & \lambda_2 & 1 & \overline{\lambda}_2 \\ 1 & \lambda_2 & \overline{\lambda}_2 & \overline{\lambda}_2 & \lambda_2 & 1 \end{pmatrix}.$$

If we denote the k^{th} row of S_3 by $(S_3)_k$, then we have

$$\begin{aligned} (S_3)_1 = (S_3)_2 = -(S_3)_3 = (S_3)_4 = -(S_3)_5 = -(S_3)_6 \\ = 1/6(1, 1, -1, 1, -1, -1). \end{aligned}$$

Also, from the general properties of S_j, we find $S_4 = S_2',\ S_5 = S_1',\ S_6 = (1)_{i,j=\overline{1,6}}$.

Now, from (15), keeping in mind (17), we can obtain what we required, i.e.,

$$\begin{aligned} &g_1^{(1)}(n,7) = 1/6[2\operatorname{Re}(\sigma_1+\sigma_2)+\sigma_3+\sigma_6], \\ &g_2^{(1)}(n,7) = 1/6[2\operatorname{Re}(\lambda_4\sigma_1+\lambda_2\sigma_2)+\sigma_3+\sigma_6], \\ &g_3^{(1)}(n,7) = 1/6[2\operatorname{Re}(\lambda_5\sigma_1+\lambda_4\sigma_2)-\sigma_3+\sigma_6], \\ &g_4^{(1)}(n,7) = 1/6[2\operatorname{Re}(\lambda_2\sigma_1+\lambda_4\sigma_2)+\sigma_3+\sigma_6], \\ &g_5^{(1)}(n,7) = 1/6[2\operatorname{Re}(\lambda_1\sigma_1+\lambda_2\sigma_2)-\sigma_3+\sigma_6], \\ &g_6^{(1)}(n,7) = 1/6[2\operatorname{Re}(-\sigma_1+\sigma_2)-\sigma_3+\sigma_6]. \end{aligned} \tag{18}$$

Since $2\lambda_2+2\lambda_4+2=0$ and $2\lambda_3+2=0$, we know the equalities obtained are true only if $n_3=n_5=0$. When $n_3\neq 0$ and $n_5=0$, we must assume that $\sigma_3=0$ in (18), but when $n_5\neq 0$ and $n_3=0$, we must assume that $\sigma_2=0$. Finally, if $n_3\neq 0$ and $n_5\neq 0$, then $\sigma_2=\sigma_3=0$. In all other cases except those indicated above, we must make use of (17).

CONCLUSION

We note here two simple properties of $g_s^{(k)}(n,p)$. Consider two rows of Pascal's triangle with numbers $(n)_p$ and $(m)_p$. First, if $(n)_p$ and $(m)_p$ contain the same figures excepting zero, then $g_s^{(k)}(n,p)=g_s^{(k)}(m,p)$ for each k and s. Second, if $(n)_p$ contains 1 ℓ more than $(m)_p$, then $g_s^{(k)}(n,p)=2^{\ell}g_s^{(k)}(m,p)$ for each k and s. The latter follows from (4) because $B_1=2E$ for each $\Delta_p^{(1)}$.

ACKNOWLEDGMENT

I wish to thank Professor Boris Bondarenko for his interest and helpful criticism and the anonymous referee for many valuable comments.

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AMS Classification Numbers: 11B65, 11B50, 11C20