A TEN POINT FFT CALCULATION WHICH FEATURES THE GOLDEN RATIO

J. M. H. Peters

The Liverpool John Moores University, Byrom Street, Liverpool L3 3AF, UK

(Submitted November 1994)

1. INTRODUCTION

One version of a discrete Fourier transform pair based on N equally spaced sample points is

$$\bar{x}_m = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi jmn}{N}}, \qquad s_m = \frac{1}{N}\sum_{n=0}^{N-1} \bar{x}_n e^{\frac{2\pi jmn}{N}}, \tag{1}$$

where $x_m = f(mT) : m = 0, 1, 2, \ldots, N-1$ for a given temporal function $f(t)$ of appropriate form, where T is the sampling interval in the time domain.

2. EXAMPLE

James et al., in [1], consider the function

$$F(t) = \begin{cases} t: & 0 \le t \le \frac{1}{2}, \\ 1-t: & \frac{1}{2} \le t \le 1, \\ 0: & t > 1, \end{cases}$$

with $N = 10$ and $T = 1/5$ sec, for which the discrete Fourier transform, computed according to (1), reduces to

$$\bar{x}_m = \frac{\left(e^{-\frac{\pi jm}{5}} + 2\left(e^{-\frac{2\pi jm}{5}} + e^{-\frac{3\pi jm}{5}}\right) + e^{-\frac{4\pi jm}{5}}\right)}{5}.$$

3. MATRIX FORMULATION

It is especially interesting, however, to give a ten point FFT analysis, where the complex exponentials are tenth roots of unity that involve the golden ratio $\tau = (1+\sqrt{5})/2$, which itself is the positive root of the quadratic equation $\tau^2 - \tau - 1 = 0$. By expressing results initially in terms of τ, rather than decimal numbers, we are able to appreciate deeper symmetries in the FFT.

By writing $\omega = e^{-\frac{\pi j}{5}} = \frac{1}{2}\left(\tau - j\sqrt{\sqrt{5}/\tau}\right)$, a tenth root of unity, the matrix representation of the first of (1) is as shown in (3) below, where the various powers of ω are, with asterisks denoting complex conjugates:

$$\omega^0 = 1,\ \omega = \frac{(\tau - j\alpha)}{2},\ \omega^2 = \frac{(1/\tau - j\alpha)}{2},\ \omega^3 = \frac{-(1/\tau + j\alpha)}{2} = -\omega^{*2},$$

$$\omega^4 = \frac{-(\tau + j\alpha)}{2} = -\omega^*,\ \omega^5 = -1,\ \omega^6 = \frac{-(\tau - j\alpha)}{2} = \omega^{*4},$$

$$\omega^7 = \frac{-(1/\tau - j\alpha)}{2} = \omega^{*3},\ \omega^8 = \frac{(1/\tau + j\alpha)}{2} = \omega^{*2},\ \omega^9 = \frac{(\tau + j\alpha)}{2} = \omega^{*}, \tag{2}$$

where $\alpha = \sqrt{\sqrt{5}/\tau}$, see [2],

$$\begin{bmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \\ \bar{x}_5 \\ \bar{x}_6 \\ \bar{x}_7 \\ \bar{x}_8 \\ \bar{x}_9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 & \omega^8 & \omega^9 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^2 & \omega^5 & \omega^8 & \omega & \omega^4 & \omega^7 \\ 1 & \omega^4 & \omega^8 & \omega^2 & \omega^6 & 1 & \omega^4 & \omega^8 & \omega^2 & 1 \\ 1 & \omega^5 & 1 & \omega^5 & 1 & \omega^5 & 1 & \omega^5 & 1 & \omega^5 \\ 1 & \omega^6 & \omega^2 & \omega^8 & \omega^4 & 1 & \omega^6 & \omega^2 & \omega^8 & \omega^4 \\ 1 & \omega^7 & \omega^4 & \omega & \omega^8 & \omega^5 & \omega^2 & \omega^9 & \omega^6 & \omega^3 \\ 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^9 & \omega^8 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \\ \bar{x}_5 \\ \bar{x}_6 \\ \bar{x}_7 \\ \bar{x}_8 \\ \bar{x}_9 \end{bmatrix}. \tag{3}$$

4. FACTORIZATION

To factor the matrix in (3), we adopt the approach used in [1], noting first that $N = 10 = 2 \times 5$ is composite, with factors $r_1 = 2$ and $r_2 = 5$. Putting

$$\begin{aligned} n &= 2n_1 + n_0: \quad n_0 = 0, 1; \qquad\qquad n_1 = 0, 1, 2, 3, 4, \\ m &= 5m_1 + m_1: \quad m_0 = 0, 1, 2, 3, 4; \quad m_1 = 0, 1, \end{aligned}$$

we can write the simultaneous system (3) as

$$\bar{x}_{5m_1+m_0} = \sum_{n_0=0}^{1} \left(\sum_{n_1=0}^{4} x_{2n_1+n_0} \omega^{2n_1 m_0} \right) \omega^{(5m_1+m_0)n_0}. \tag{4}$$

Setting

$$\xi_{m_0 n_0} = \sum_{n_1=0}^{4} x_{2n_1+n_0} \omega^{2n_1 m_0} : n_0 = 0, 1;\ m_0 = 0, 1, 2, 3, 4, \tag{5}$$

then leads to a set of simultaneous equations, summarized in matrix form by

$$\begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \xi_{10} \\ \xi_{11} \\ \xi_{20} \\ \xi_{21} \\ \xi_{30} \\ \xi_{31} \\ \xi_{40} \\ \xi_{41} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & \omega^2 & 0 & \omega^4 & 0 & \omega^6 & 0 & \omega^8 & 0 \\ 0 & 1 & 0 & \omega^2 & 0 & \omega^4 & 0 & \omega^6 & 0 & \omega^8 \\ 1 & 0 & \omega^4 & 0 & \omega^8 & 0 & \omega^2 & 0 & \omega^6 & 0 \\ 0 & 1 & 0 & \omega^4 & 0 & \omega^8 & 0 & \omega^2 & 0 & \omega^6 \\ 1 & 0 & \omega^6 & 0 & \omega^2 & 0 & \omega^8 & 0 & \omega^4 & 0 \\ 0 & 1 & 0 & \omega^6 & 0 & \omega^2 & 0 & \omega^8 & 0 & \omega^4 \\ 1 & 0 & \omega^8 & 0 & \omega^6 & 0 & \omega^4 & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & \omega^8 & 0 & \omega^6 & 0 & \omega^4 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}. \tag{6}$$

5. SOLUTIONS

Inserting the appropriate powers of ω, summarized in (2), into the linear system (6), leads to the following results:

$$\xi_{00} = 3/5 = \xi_{01}, \qquad \xi_{10} = \frac{\left(-\frac{1}{\tau^2} - j\sqrt{\sqrt{5}\tau^5}\right)}{10}, \qquad \xi_{11} = \frac{\left(\tau - j\sqrt{\sqrt{5}\tau}\right)}{5},$$

$$\xi_{20} = \frac{\left(-\tau^2 - j\sqrt{\sqrt{5}/\tau^5}\right)}{10}, \qquad \xi_{21} = \frac{\left(-\frac{1}{\tau} - j\sqrt{\sqrt{5}/\tau}\right)}{5}, \qquad \xi_{30} = \frac{\left(-\tau^2 + j\sqrt{\sqrt{5}/\tau^5}\right)}{10} = \bar{\xi}_{20},$$

$$\xi_{31} = \frac{\left(-\frac{1}{\tau} + j\sqrt{\sqrt{5}/\tau}\right)}{5} = \bar{\xi}_{21}, \quad \xi_{40} = \frac{\left(-\frac{1}{\tau^2} + j\sqrt{\sqrt{5}\tau^5}\right)}{10} = \bar{\xi}_{10}, \quad \xi_{41} = \frac{\left(\tau + j\sqrt{\sqrt{5}\tau}\right)}{5} = \bar{\xi}_{11}.$$

Returning to the system (4) we see that, with (5), we can write

$$\bar{x}_{5m_1+m_0} = \sum_{n_0=0}^{1} \xi_{m_0 n_0} \omega^{(5m_1+m_0)n_0} : \ m_0 = 0, 1, 2, 3, 4; \ m_1 = 0, 1.$$

6. NUMERICAL RESULTS

Expansion leads to

$$\begin{aligned}
\bar{x}_0 &= \xi_{00} + \xi_{01} = 6/5 = 1.2, \\
\bar{x}_1 &= \xi_{10} + \xi_{11}\omega = -j\sqrt{\sqrt{5}\tau^5}/5 = -j0.9959593, \\
\bar{x}_2 &= \xi_{20} + \xi_{21}\omega^2 = -\tau^2/5 = -0.5236068, \\
\bar{x}_3 &= \xi_{30} + \xi_{31}\omega^3 = j\sqrt{\sqrt{5}/t^5} = j0.0898055, \\
\bar{x}_4 &= \xi_{40} + \xi_{41}\omega^4 = -(1/\tau^2)/5 = -0.076392, \\
\bar{x}_5 &= \xi_{00} + \xi_{01}\omega^5 = 0, \\
\bar{x}_6 &= \xi_{10} + \xi_{11}\omega^6 = \bar{x}_4^* = -(1/\tau^2)/5 = -0.076392, \\
\bar{x}_7 &= \xi_{20} + \xi_{21}\omega^7 = \bar{x}_3^* = -j\sqrt{\sqrt{5}/\tau^5} = -j0.0898055, \\
\bar{x}_8 &= \xi_{30} + \xi_{31}\omega^8 = \bar{x}_2^* = -\tau^5/5 = -0.5236068, \\
\bar{x}_9 &= \xi_{40} + \xi_{41}\omega^9 = \bar{x}_1^* = j\sqrt{\sqrt{5}t^5}/5 = j0.9959593.
\end{aligned}$$

Multiplying each of these by $T = 0.2$ gives James et al.'s final results (see [1]).

REFERENCES

1. G. James et al. *Advanced Modern Engineering Mathematics.* New York: Addison-Wesley, 1993.
2. Marjorie Bicknell & Verner E. Hoggatt, Jr. (eds.). *A Primer for the Fibonacci Numbers*, pp. 74-75. Santa Clara, CA: The Fibonacci Association, 1972.

AMS Classification Numbers: 11B39, 15A23, 65T20

❖❖❖