ON THE ZECKENDORF FORM OF F_{kn} / F_n

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1. INTRODUCTION

Filipponi and Freitag [1] obtained the Zeckendorf representation F_{kn} / F_n for $k, n \geq 1$ and showed that its form depends on the parity of n and the congruence class of k modulo 4. These representations were not deduced directly, but were conjectured from numerical evidence. The purpose of this note is to present a constructive proof of these results. An important intermediate step in our proof, which is of interest in its own right, is to obtain the Zeckendorf form for the difference of two Fibonacci numbers.

2. A SUBTRACTION ALGORITHM

Theorem: For $n, k \geq 1$,

$$F_{n+k} - F_n = \sum_{r=1}^{[k/2]} F_{n+k+1-2r} + \begin{cases} 0, & k \text{ even}, \\ F_{n-1}, & k \text{ odd}, \end{cases} \tag{1}$$

where $[x]$ denotes the greatest integer not greater than x and, when $k = 1$, the empty sum denotes zero.

Proof: We will show by induction on k that (1) holds for $k \geq 1$ and all $n \geq 1$. First we note that (1) holds for $k = 1$ and for $k = 2$ and all $n \geq 1$. We now write

$$F_{n+k+2} - F_n = F_{n+k+1} + (F_{n+k} - F_n).$$

Thus, from (1),

$$F_{n+k+2} - F_n = F_{n+k+1} + \sum_{r=1}^{[k/2]} F_{n+k+1-2r} + \begin{cases} 0, & k \text{ even}, \\ F_{n-1}, & k \text{ odd}, \end{cases}$$

and hence

$$F_{n+k+2} - F_n = \sum_{r=1}^{[k/2]+1} F_{n+k+3-2r} + \begin{cases} 0, & k \text{ even}, \\ F_{n-1}, & k \text{ odd}. \end{cases}$$

We have shown that if (1) holds for some $k \geq 1$ it also holds for $k + 2$ and, since (1) holds for $k = 1$ and $k = 2$, it holds for all $k \geq 1$. □

3. THE MAIN RESULT

Let us replace F_{kn} and F_n by their Binet forms to give

$$\frac{F_{kn}}{F_n} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha^n - \beta^n} = \sum_{s=1}^{k} \alpha^{(k-s)n}\beta^{(s-1)n}. \tag{2}$$

We may combine the terms of the latter sum in pairs and use the relation $\alpha\beta = -1$, writing

$$\alpha^{(k-r)n}\beta^{(r-1)n} + \alpha^{(r-1)n}\beta^{(k-r)n} = (-1)^{(r-1)n}(\alpha^{(k-2r+1)n} + \beta^{(k-2r+1)n}) = (-1)^{(r-1)n}L_{(k-2r+1)n}$$

for $1 \le r \le [k/2]$, to express the right side of (2) in terms of Lucas numbers. The result clearly depends on the parity of k, and we obtain

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{[k/2]} (-1)^{(r-1)n} L_{(k-2r+1)n} + \begin{cases} (-1)^{(k-1)n/2}, & k \text{ odd}, \\ 0, & k \text{ even}, \end{cases} \tag{3}$$

for $k \ge 2$. This combines two formulas quoted in Vajda [2, p. 182]. We now derive the Zeckendorf form of F_{kn}/F_n from (3) as follows. First, for n even we have, on replacing each L_m in (3) by $F_{m+1} + F_{m-1}$,

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{[k/2]} (F_{(k-2r+1)n+1} + F_{(k-2r+1)n-1}) + \begin{cases} F_2, & k \text{ odd}, \\ 0, & k \text{ even}, \end{cases} \tag{4}$$

which is in Zeckendorf form. (See Filipponi & Freitag [1, formulas (2.1) and (2.2)].)

For n odd, more effort is required. First we obtain from (3) that

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{[k/2]} (-1)^{(r-1)} L_{(k-2r+1)n} + \begin{cases} (-1)^{(k-1)/2}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \tag{5}$$

Because of the alternating signs in (5), we need to group the terms in pairs, as far as possible. With $k \ge 4$, the first pair in (5) is

$$L_{(k-1)n} - L_{(k-3)n} = F_{(k-1)n+1} + F_{(k-1)n-1} - F_{(k-3)n+1} - F_{(k-3)n-1}. \tag{6}$$

On using the above "subtraction algorithm" to combine the second and third terms on the right of (6), we derive

$$L_{(k-1)n} - L_{(k-3)n} = F_{(k-1)n+1} + \left(\sum_{s=1}^{n-1} F_{(k-1)n-2s}\right) - F_{(k-3)n-1},$$

from which we obtain the Zeckendorf form

$$L_{(k-1)n} - L_{(k-3)n} = F_{(k-1)n+1} + \left(\sum_{s=1}^{n-2} F_{(k-1)n-2s}\right) + F_{(k-3)n+1} + F_{(k-3)n-2}.$$

For a general pair of terms on the right of (5), we have the Zeckendorf form

$$L_{(k-4r-1)n} - L_{(k-4r-3)n} = F_{(k-4r-1)n+1} + \left(\sum_{s=1}^{n-2} F_{(k-4r-1)n-2s}\right) + F_{(k-4r-3)n+1} + F_{(k-4r-3)n-2}.$$

Thus, for n odd, it is clear that the transformation of (5) into Zeckendorf form depends on whether $[k/2]$ is odd or even; that is, the final form depends on the residue class of k modulo 4. First, for $k = 4m$ and n odd, we obtain

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{m} (L_{(k-4r+3)n} - L_{(k-4r+1)n})$$

and thus

$$\frac{F_{kn}}{F_n} = S_{k,n}, \tag{7}$$

say, where

$$S_{k,n} = \sum_{r=0}^{[k/4]-1} \left(F_{(k-4r-1)n+1} + \left(\sum_{s=1}^{n-2} F_{(k-4r-1)n-2s} \right) + F_{(k-4r-3)n+1} + F_{(k-4r-3)n-2} \right). \tag{8}$$

We similarly work through the other cases, where n is odd and $k \equiv 1, 2$, and 3 (mod 4). In each case, the "most significant" part of the Zeckendorf form is $S_{k,n}$, defined by (8). The precise Zeckendorf form is

$$\frac{F_{kn}}{F_n} = S_{k,n} + e_{k,n}, \tag{9}$$

where the least significant part of the Zeckendorf sum is

$$e_{k,n} = \begin{cases} 0, & k \equiv 0 \pmod 4, \\ F_2, & k \equiv 1 \pmod 4, \\ F_{n+1} + F_{n-1}, & k \equiv 2 \pmod 4, \\ F_{2n+1} + \sum_{r=1}^{n-1} F_{2n-2r}, & k \equiv 3 \pmod 4. \end{cases} \tag{10}$$

Thus the Zeckendorf representation of F_{kn} / F_n is given by (4) for n even and by (9) and (10) for n odd.

Note added in proof: The relation (1) above appeared earlier in Filipponi [3].

REFERENCES

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