ON TRIANGULAR AND BAKER'S MAPS WITH GOLDEN MEAN AS THE PARAMETER VALUE

Chyi-Lung Lin
Department of Physics, Soochow University, Taipei, Taiwan 111, R.O.C.
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1. INTRODUCTION

We discuss the triangular map and also Baker's map [2], [4] with the parameter μ chosen with the value of the golden mean: $(1+\sqrt{5})/2 \approx 1.618$. For an arbitrary parameter value in the range $0 \le \mu \le 2$, the starting graph $(x, f(x))$ in the range $0 \le x \le 1$ is a line segment for a triangular map, and two line segments for Baker's map (see Fig. 1 and Fig. 4). We are interested in the graph of $f^{[n]}$, where $n > 1$. Since the starting graph contains a set of line segments, the proceeding graphs $(x, f^{[n]}(x))$ are then obtained from iterates of the beginning line segments in the starting graph. It is therefore important to discuss iterating these two maps on line segments. Since these two maps are piecewise linear maps, it can then be shown that the graph of $f^{[n]}$ is a composition of line segments (see Fig. 3 and Fig. 6). These two maps are simple for μ in the range $0 \le \mu \le 1$ because the number of line segments does not increase under the action of mapping; the graph of $f^{[n]}$ is therefore simple. Yet, they are often complicated in the range $\mu > 1$, as the action of mapping on starting line segments will generate more line segments of which the lengths in general are different. The graph of $f^{[n]}$ is then a set of line segments with irregular shape which becomes very complicated when n is large. It is then difficult to determine the graph of $f^{[n]}$. However, we can show that when μ is chosen with specific values, for instance, the golden mean, the graphs $(x, f^{[n]}(x))$ are again simple. There are only a few types of line segments in each graph and, interestingly, the numbers of line segments of the graphs are those of the Fibonacci numbers. Nature shows that Fibonacci numbers occur quite frequently in various areas; therefore, it is interesting to know that Fibonacci numbers and, in fact, Fibonacci numbers of degree m, can be generated from a simple dynamical system [3]. In this work, we contain some reviews of [3], show the similarity of these two maps when a specific parameter value is chosen, derive geometrically a well-known identity in Fibonacci numbers, and show that some invariant sequences can be obtained.

2. THE TRIANGULAR MAP WITH $\mu = (1+\sqrt{5})/2$

First, we discuss the general triangular T_μ map, which is defined by

$$T_\mu(x) = 1 - \mu|x|, \tag{1}$$

or

$$x_{n+1} = 1 - \mu|x_n|, \tag{2}$$

where μ is the parameter. We restrict the ranges to: $-1 \le x \le 1$ and $0 \le \mu \le 2$, so that T_μ maps from the interval $[-1, 1]$ to $[-1, 1]$. Figure 1 shows three graphs of T_μ for, respectively, $\mu = 0.6, 1,$

and $(1+\sqrt{5})/2$. We define $x_1 \equiv \mathrm{T}_\mu(x)$ as the first iterate of x for T_μ, and $x_n = \mathrm{T}_\mu(x_{n-1}) \equiv \mathrm{T}_\mu^{[n]}(x)$ as the n^{th} iterate of x for T_μ. Since all the graphs $(x, \mathrm{T}_\mu^{[n]}(x))$ are symmetrical for $x>0$ and $x<0$, we henceforth consider these graphs in only the region of $x \geq 0$. The starting graph, $\mathrm{T}_\mu(x) = 1-\mu x$, in the range $x \geq 0$ is then a line segment from point $(0, 1)$ to point $(1, 1-\mu)$ with slope $-\mu$. We call this the **starting line segment**. Iterating this map on this starting line segment then generates all the proceeding $\mathrm{T}_\mu^{[n]}$ graphs.

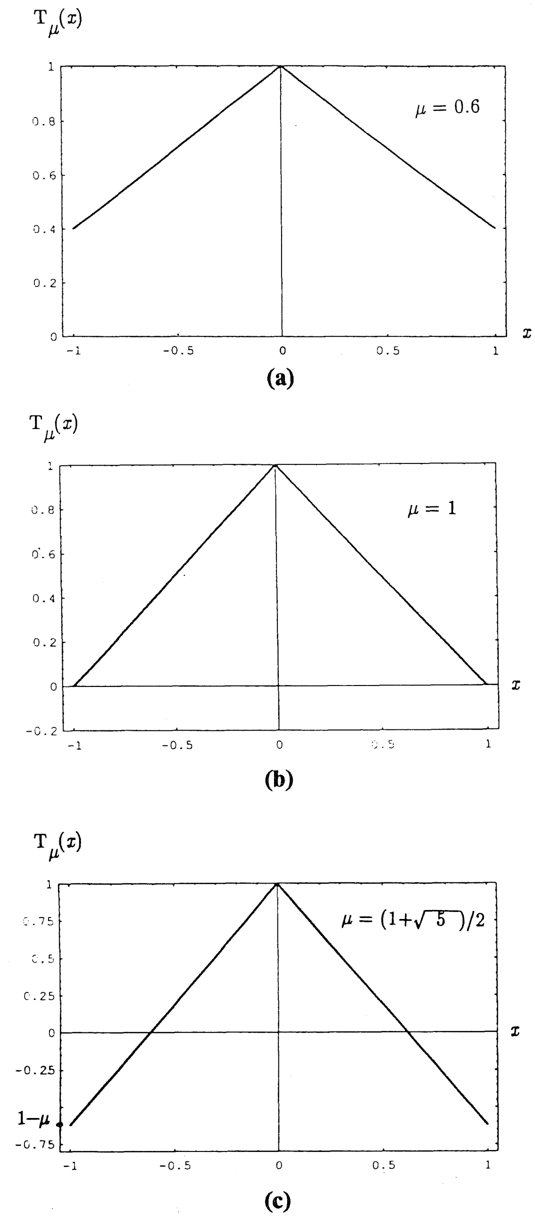

FIGURE 1

For a general discussion on an iterate of a line segment, we consider an arbitrary line segment described by $y = g(x) = a + bx$, where $0 \le x \le 1$ and $-1 \le g(x) \le 1$. The upper piece of this line segment, for which $g(x) \ge 0$, after one iteration, goes to a line segment described by $y = \mathrm{T}_\mu(g(x)) = 1 - a\mu - b\mu x$. Hence, the slope and the length of the upper piece have been rescaled, the sign of the slope alters as well; the slope rescaling factor is seen to be $-\mu$. The lower piece, for which $g(x) \le 0$, goes to another line segment described by $y = \mathrm{T}_\mu(g(x)) = 1 + a\mu + b\mu x$. Hence, the slope and the length of the lower piece have been rescaled, the sign of the slope is not altered; the slope rescaling factor is μ. Since the slope rescaling factors are different for these two pieces, a line segment after one iteration will be folded into two connected line segments if this line segment intersects the x-axis. A line segment that does not intersect the x-axis will only change its slope and length but not be folded. These are useful in graphical analysis of iterates.

For μ in the range $0 \le \mu < 1$, the T_μ map is simple, as there is one stable fixed point $x^* = 1/(1+\mu)$, and the basin of attraction of x^* consists of all $x \in [-1, 1]$, hence the graph of $\mathrm{T}_\mu^{[n]}$, as $n \to \infty$, will finally approach a flat line segment with height x^*. Or, we may see this from iterates of the starting line segment. Since the starting line segment does not intersect the x-axis and the slope rescaling factor is $-\mu$ for which $|\mu| < 1$, the starting line segment under iterations then remains a line segment and getting flatter but with one point fixed. The graph of $\mathrm{T}_\mu^{[n]}$, as $n \to \infty$, then approaches a flat line segment with a height which can only be of the value of the fixed point, that is, x^*. This map with $\mu < 1$ is therefore simple.

For $\mu = 1$, we have $\mathrm{T}_\mu(x) = 1 - x$ and $\mathrm{T}_\mu^{[2]}(x) = x$, hence $\{x, 1-x\}$ is a 2-cycle for each x, and we have, therefore, only two shapes which are the graphs $(x, \mathrm{T}_\mu(x))$ and $(x, \mathrm{T}_\mu^{[2]}(x))$. Or, since the starting line segment, with slope -1, does not pass through the x-axis and the slope rescaling factor is -1; therefore, the starting line segment under iterations remains a line segment but with slope 1 and -1, alternatively. This map with $\mu = 1$ is also a simple map.

For $1 < \mu \le 2$, there exist no stable fixed points for T_μ, $\mathrm{T}_\mu^{[2]}$, and in fact for $\mathrm{T}_\mu^{[n]}$ with n any positive integer. This is because the |slope| at each fixed point of $\mathrm{T}_\mu^{[n]}$ equals μ^n which is greater than one; hence, there are no stable fixed points for $\mathrm{T}_\mu^{[n]}$. In such a region without any stable cycles, the iterative behaviors in general are complicated. Indeed, as the starting line segment now intersects the x-axis, the action of mapping will keep on folding line segments and, therefore, producing more and more line segments of which the lengths in general are different; thus, the line segments in each graph are of many types. Each graph is then the connection of line segments of many types and, therefore, has an irregular zigzag shape. The complication of the graph $\mathrm{T}_\mu^{[n]}$ increases with n, and it is hard to predict what the final result will be. Interestingly, there are cases that are easier to analyze. We may consider focusing on some particular values of μ such that the iterative behaviors are simple.

To choose the proper values of μ in the range $1 < \mu \le 2$, the consideration is upon the particular point of $x = 0$ at which the function $\mathrm{T}_\mu(x)$ has a maximum height of 1. We require that this point be a periodic point of the map. The orbit of $x = 0$ is 0, $\mathrm{T}_\mu(0)$, $\mathrm{T}_\mu^{[2]}(0)$, $\mathrm{T}_\mu^{[3]}(0)$, ... etc. We can easily see that $\mathrm{T}_\mu^{[2]}(0) = 1 - \mu$ and $\mathrm{T}_\mu^{[3]}(0) = 1 - \mu(\mu - 1) = 1 + \mu - \mu^2$. The required parameter value for $x = 0$ becoming a period-2 point is determined from the equation $\mathrm{T}_\mu^{[2]}(0) = 0$. The solution is $\mu = 1$. As discussed above, it is a simple map in this case. Requiring $x = 0$ to be a period-3 point, we should have $\mathrm{T}_\mu^{[3]}(0) = 0$; that is, $\mu^2 - \mu - 1 = 0$. For $\mu > 1$, the solution is $\mu = (1 + \sqrt{5})/2$, which is the well-known golden mean. With these basic arguments, we then have the following results when $\mu=(1+\sqrt{5})/2$.

Proposition 2.1: The point $x=0$ is a period-3 point of the triangular T_μ map.

Proof: This is obvious, as we see that the 3-cycle is $\{0,1,1-\mu\}$; $1-\mu=-1/\mu$.

In what follows, we will discuss the graph $y=\mathrm{T}_\mu^{[n]}(x)$ in the range $0\le x\le 1$. It is important to discuss an iterate of a line segment. We first define two types of line segments. We denote by L a **long line segment** connecting points $(x_1, 1-\mu)$ and $(x_2, 1)$, where $0\le x_1,\ x_2\le 1$ and by S a **short line segment** connecting points $(x_3, 0)$ and $(x_4, 1)$, where $0\le x_3,\ x_4\le 1$. The four graphs of Figure 2 show some examples of line segments of these two types, where the subscripts + and – label line segments with positive and negative slope, respectively.

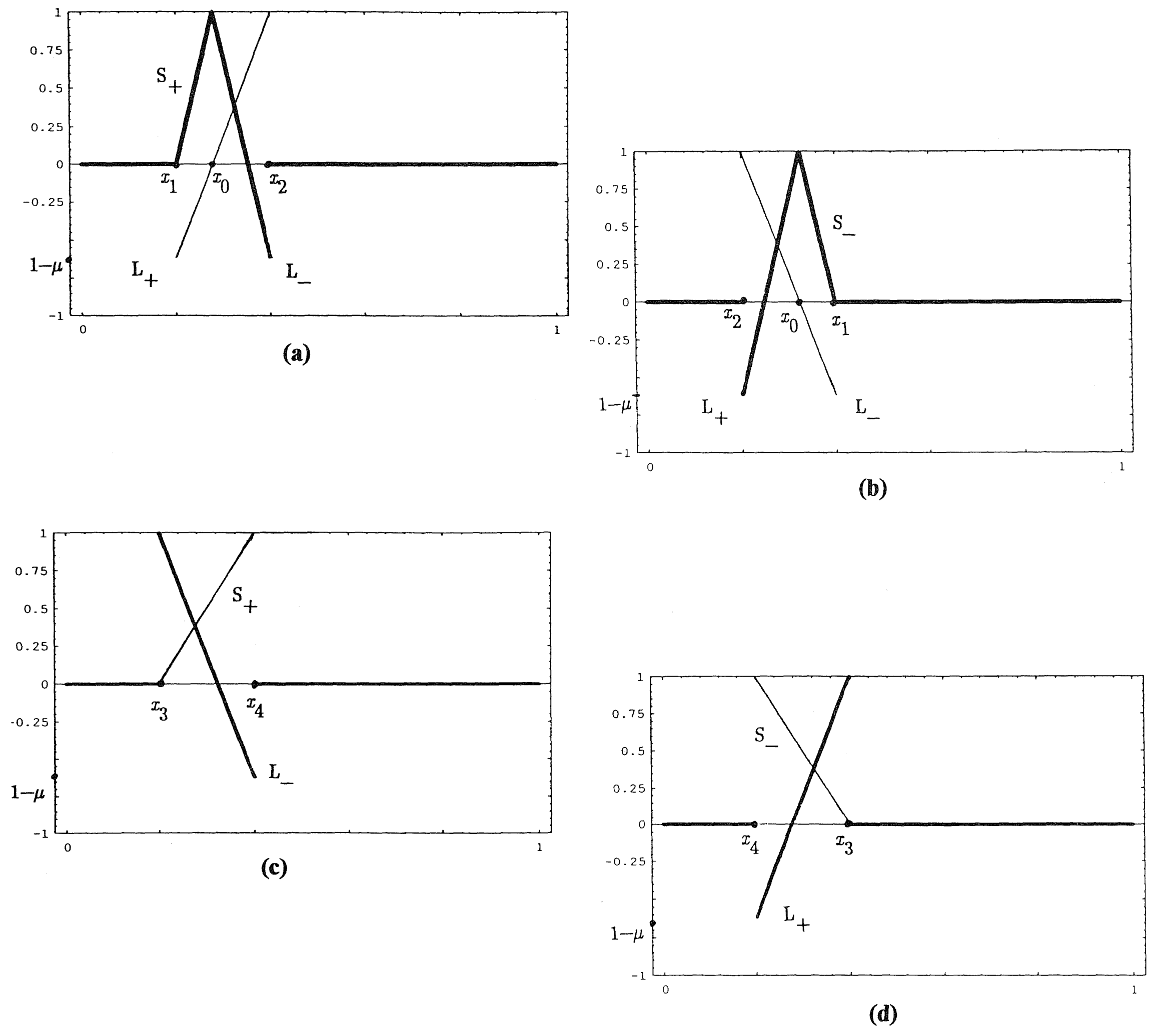

FIGURE 2

In fact, as will be shown, we need only consider an iterate of line segments of type L and type S; hence, we now discuss $T_\mu(L(x))$ and $T_\mu(S(x))$, where $L(x)$ and $S(x)$ are linear functions whose graphs are **long** and **short** segments. Using $T_\mu(0)=1$, $T_\mu(1)=1-\mu$, and $T_\mu(1-\mu)=0$, Figure 2(a) then shows that a line segment L_+ connecting points $(x_1, 1-\mu)$ and $(x_2, 1)$, after one iteration, is folded into two line segments, of which one connects points $(x_1, 0)$ and $(x_0, 1)$, i.e., an S_+, and the other connects points $(x_0, 1)$ and $(x_2, 1-\mu)$, i.e., an L_-, where $(x_0, 0)$ is the intersection point of the line segment L_+ with the x-axis. Therefore, an L_+ after one iteration goes to an S_+ and an L_-, we denote this by $T_\mu: L_+ \to S_+L_-$. The graphs of Figure 2(b), 2(c), and 2(d) show an iterate of a line segment of type L_-, S_+, and S_-, respectively. From these, we conclude that the action of this map on line segments of these four types is described as:

$$\begin{aligned} T_\mu: L_+ &\to S_+L_- \\ L_- &\to L_+S_- \\ S_+ &\to L_- \\ S_- &\to L_+ \end{aligned} \tag{3}$$

T_μ then acts as a discrete map for L and S. If we are only to count the number of line segments in a graph—we need not distinguish L_+ from L_-, S_+ from S_-, and LS from SL—then (3) can be expressed more simply as:

$$\begin{aligned} T_\mu: L &\to LS \\ S &\to L \end{aligned} \tag{4}$$

From (4), we have the following results.

Proposition 2.2: The graph of $y = T_\mu^{[n]}(x)$ in the range $0 \le x \le 1$ contains line segments of only two types, type L and type S, and the total number of these line segments is F_{n+1}.

Proof: Since the starting graph is a line segment connecting points $(0, 1)$ and $(1, 1-\mu)$ [see Fig. 1(c)], it is thus a line segment of type L. From (4), we see that, starting from an L, line segments generated from iterates of that are, therefore, only of two types: type L and type S. From (4), we also see that line segments of type L and S, respectively, are similar to those rabbits of type large and small in the original Fibonacci problem; hence, the numbers of line segments of all the graphs $T_\mu^{[n]}$ would be those of the Fibonacci numbers. Therefore, we have shown an interpretation of the Fibonacci sequence from the point of view of a simple iterated map. Although we start from a functional map T_μ on a finite interval of x, however, if we take line segments as the entities, then T_μ acts as a discrete map for these entities, and the mechanism of generating line segments from the action of this discrete map is now the same as the breeding of the Fibonacci rabbits. We now let $L(n)$ and $S(n)$ represent, respectively, the numbers of L's and S's in the graph of $T_\mu^{[n]}$. Then, from (4), we have

$$\begin{aligned} L(n) &= L(n-1) + S(n-1), \\ S(n) &= L(n-1). \end{aligned} \tag{5}$$

Equation (5) shows that $L(n) = L(n-1) + L(n-2)$ and $S(n) = S(n-1) + S(n-2)$. Thus, both sequences $\{L(n)\}$ and $\{S(n)\}$ are the Fibonacci-type sequences. Since we start from an L with slope $-\mu$, we have $\mathrm{L}(1)=1$ and $\mathrm{S}(1)=0$. According to (5), we then have $\mathrm{L}(n)=F_n$ and $\mathrm{S}(n)=F_{n-1}$, where F_n is the n^{th} Fibonacci number, and the |slope| of each line segment in the graph of $\mathrm{T}_\mu^{[n]}$ is μ^n. Therefore, the total number of line segments in the graph of $\mathrm{T}_\mu^{[n]}$ is $\mathrm{L}(n)+\mathrm{S}(n)=F_n+F_{n-1}=F_{n+1}$. Figure 3 shows the graph of $\mathrm{T}_\mu^{[5]}$, from which we can count the number of line segments as being $F_6=8$.

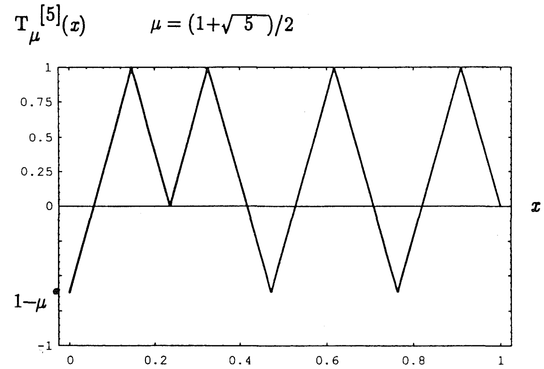

FIGURE 3

In [1] there is an interesting theorem stating that $\lim_{n\to\infty}\log C(n,\mu)/n=\log\mu$, where $C(n,\mu)$ is the number of line segments of the graph $y=f^{[n]}(x,\mu)$ and μ is an arbitrary parameter value. In our case, $C(n,\mu)=F_{n+1}$ and $\mu=(1+\sqrt{5})/2$, so using the well-known formula $F_n=(\mu^n-(1-\mu)^n)/\sqrt{5}$ we can easily calculate that $\lim_{n\to\infty}\log C(n,\mu)/n$ is indeed the value $\log\mu$.

Proposition 2.3: A simple identity, $\mu F_n+F_{n-1}=\mu^n$.

Proof: In the graph of $\mathrm{T}_\mu^{[n]}$, we have F_n long line segments and F_{n-1} short line segments, and the |slope| of each line segment is μ^n. We denote by $d(\mathrm{L},n)$ and $d(\mathrm{S},n)$, respectively, the projection length of a line segment of type L and type S on the x-axis in the graph of $\mathrm{T}_\mu^{[n]}$. We then have $d(\mathrm{L},n)=(\mu)^{1-n}$ and $d(\mathrm{S},n)=(\mu)^{-n}$. Since the total projection length of these F_{n+1} line segments in the x-axis should be 1, we have

$$F_n d(\mathrm{L},n)+F_{n-1}d(\mathrm{S},n)=1 \tag{6}$$

or

$$\mu F_n+F_{n-1}=\mu^n. \tag{7}$$

This is a well-known identity in Fibonacci numbers; here we have derived it geometrically.

Proposition 2.4: There are three infinite LS sequences that are invariant under three iterations of the map (3).

Proof: We consider the shape of the graph of $y=\mathrm{T}_\mu^{[n]}(x)$ in the range $0\le x\le 1$. Since all the shapes are from the connection of line segments of type L_+, L_-, S_+, and S_-; therefore, we can describe each shape in terms of an LS sequence. We let $\mathrm{LS}[\mathrm{T}^n]$ represent the LS sequence describing the shape of the graph of $y = \mathrm{T}_\mu^{[n]}(x)$ in the range $0 \le x \le 1$. The starting graph is simply L_-. From (3), we have:

$$\begin{aligned}
&\mathrm{LS}[\mathrm{T}^1] = \mathrm{L}_- \\
&\mathrm{LS}[\mathrm{T}^2] = \mathrm{L}_+\mathrm{S}_- \\
&\mathrm{LS}[\mathrm{T}^3] = \mathrm{S}_+\mathrm{L}_-\mathrm{L}_+ \\
&\mathrm{LS}[\mathrm{T}^4] = \mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_- \\
&\mathrm{LS}[\mathrm{T}^5] = \mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_- \\
&\mathrm{LS}[\mathrm{T}^6] = \mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+ \\
&\dots \\
&\mathrm{LS}[\mathrm{T}^n] = \mathrm{LS}[\mathrm{T}^{n-3}]\mathrm{LS}[\mathrm{T}^{n-2}]\mathrm{LS}[\mathrm{T}^{n-4}]\mathrm{LS}[\mathrm{T}^{n-3}]
\end{aligned} \tag{8}$$

The last formula in (8) is for general n and can be proved easily by induction: starting from $n=5$, we have $\mathrm{LS}[\mathrm{T}^5] = \mathrm{LS}[\mathrm{T}^2]\mathrm{LS}[\mathrm{T}^3]\mathrm{LS}[\mathrm{T}^1]\mathrm{LS}[\mathrm{T}^2]$, and then, after one iteration, we have $\mathrm{LS}[\mathrm{T}^6] = \mathrm{LS}[\mathrm{T}^3]\mathrm{LS}[\mathrm{T}^4]\mathrm{LS}[\mathrm{T}^2]\mathrm{LS}[\mathrm{T}^3]$, ... etc. The length of the $\mathrm{LS}[\mathrm{T}^n]$ sequence is F_{n+1}. Equation (8) shows that, after three iterations, an LS sequence will get longer but the original LS sequence remains. By taking $n \to \infty$, we then have an infinite LS sequence which is invariant after three iterations of the map (3), since from (8) we have

$$\lim_{n\to\infty} \mathrm{LS}[\mathrm{T}^n] = \lim_{n\to\infty} \mathrm{LS}[\mathrm{T}^{n-3}].$$

We denote by $\{\mathrm{T}_1^\infty\}$ the first invariant infinite LS sequence obtained from iterates of an L_-, that is,

$$\{\mathrm{T}_1^\infty\} = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{L}_-).$$

We see that $\{\mathrm{T}_1^\infty\}$ is invariant after three iterations of the map (3). There are two other invariant infinite sequences which we denote by $\{\mathrm{T}_2^\infty\}$ and $\{\mathrm{T}_3^\infty\}$, where $\{\mathrm{T}_2^\infty\}$ is obtained from an iterate of $\{\mathrm{T}_1^\infty\}$, i.e., $\{\mathrm{T}_2^\infty\} = \mathrm{T}_\mu\{\mathrm{T}_1^\infty\}$, and $\{\mathrm{T}_3^\infty\}$ is from an iterate of $\{\mathrm{T}_2^\infty\}$, i.e., $\{\mathrm{T}_3^\infty\} = \mathrm{T}_\mu\{\mathrm{T}_2^\infty\} = \mathrm{T}_\mu^{[2]}\{\mathrm{T}_1^\infty\}$. Therefore, there are at least three infinite LS sequences that are invariant after three iterations of the map (3). Since

$$\{\mathrm{T}_2^\infty\} = \mathrm{T}_\mu\{\mathrm{T}_1^\infty\} = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n+1]}(\mathrm{L}_-) = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{L}_+\mathrm{S}_-) = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{L}_+),$$

this means that $\{\mathrm{T}_2^\infty\}$ can be obtained from iterates of an L_+. Finally, since

$$\{\mathrm{T}_3^\infty\} = \mathrm{T}_\mu\{\mathrm{T}_2^\infty\} = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n+1]}(\mathrm{L}_+) = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{S}_+\mathrm{L}_-) = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{S}_+),$$

this means that $\{\mathrm{T}_3^\infty\}$ can be obtained from iterates of an S_+. Therefore, from the first few iterates of L_+, L_-, and S_+, we have the following first few elements of these three infinite sequences:

$$\begin{aligned}
\{\mathrm{T}_1^\infty\} &= \mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_- \dots \\
\{\mathrm{T}_2^\infty\} &= \mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+ \dots \\
\{\mathrm{T}_3^\infty\} &= \mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+ \dots
\end{aligned} \tag{9}$$

There is no invariant sequence with S_- as the first element since, after three iterations, S_- goes to $\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-$, the first element is now L_- instead of S_-. If a sequence were an invariant sequence, its length must be infinite and its first element can only be an L_+ or an L_- or an S_+, and that would just correspond to the infinite sequences $\{T_1^\infty\}$, $\{T_2^\infty\}$, and $\{T_3^\infty\}$; hence, there are only three infinite LS sequences that are invariant after three iterations of the map (3). As a result, to arrange LS sequences that are invariant after three iterations of the map (3), the L_+'s, L_-'s, S_+'s, and S_-'s should be arranged according to the orders described in (9). This interesting phenomenon may have applications in genetics. The general rule for deciding the n^{th} entry in each of these three sequences is complicated; we will discuss this in the next section, where we treat a similar but simpler case.

3. BAKER'S MAP WITH $\mu=(1+\sqrt{5})/2$

We now consider the easier Baker B_μ map, which is defined by

$$B_\mu(x)=\begin{cases}\mu x & \text{for } 0\le x\le 1/2,\\ \mu(x-1/2) & \text{for } 1/2<x\le 1,\end{cases} \tag{10}$$

where μ is the parameter. We restrict the ranges to $0\le\mu\le 2$ and $0\le x\le 1$, so that B_μ maps from the interval [0, 1] to [0, 1]. Figure 4 shows the graph of B_μ for $\mu=(1+\sqrt{5})/2$.

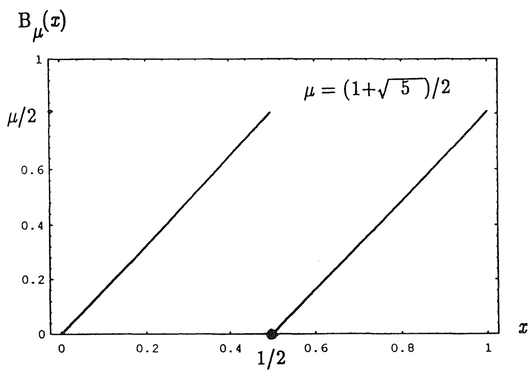

FIGURE 4

The interesting point of x now is $x=1/2$. We require that $x=1/2$ be a period-2 point of the map. The parameter value in this condition is easily determined to be $\mu=(1+\sqrt{5})/2$. Hence, μ is again the value of the golden mean. Using this parameter value, we have the following results.

Proposition 3.1: The point $x=1/2$ is a period-2 point of the B_μ map.

Proof: We easily see that the 2-cycle is $\{1/2, \mu/2\}$ when $\mu=(1+\sqrt{5})/2$.

The starting graph in the range $0\le x\le 1$ is $y=B_\mu(x)$ (see Fig. 4). It consists of two parallel line segments: one connecting points $(0,0)$ and $(1/2,\mu/2)$; the other connecting points $(1/2,0)$ and $(1,\mu/2)$. We denote by L a **long line** segment connecting points $(x_1,0)$ and $(x_2,\mu/2)$, where $x_2>x_1$. and by S a **short line** segment connecting points $(x_3,0)$ and $(x_4,1/2)$, where $x_4>x_3$. Figure 5 shows some examples of line segments of these two types.

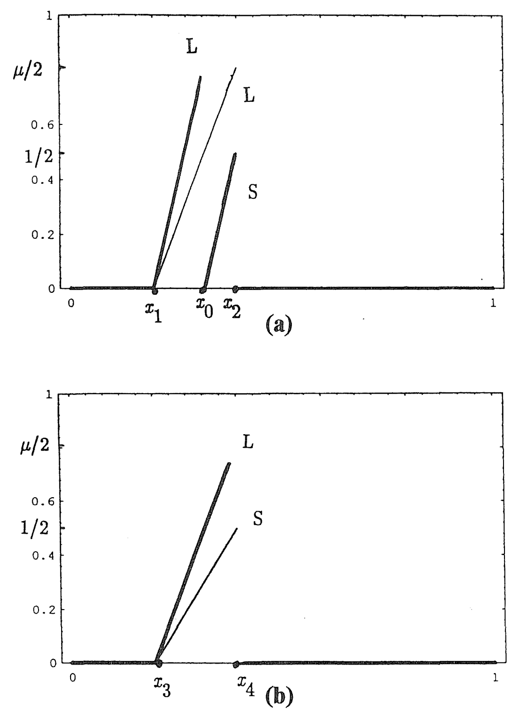

FIGURE 5

The graph of $y = \mathrm{B}_\mu(x)$ then contains two parallel line segments of type L with slope μ. Using $\mathrm{B}_\mu(0) = 0$, $\mathrm{B}_\mu(1/2) = \mu/2$, and $\mathrm{B}(\mu/2) = 1/2$, Figure 5(a) shows that a long line segment connecting points $(x_1, 0)$ and $(x_2, \mu/2)$, after one iteration, splits into two line segments, of which one is a long segment connecting points $(x_1, 0)$ and $(x_2, \mu/2)$ and the other is a short segment connecting points $(x_0, 0)$ and $(x_2, 1/2)$, where $x_1 < x_0 < x_2$ and $\mathrm{B}_\mu(x_0) = 1/2$. Figure 5(b) shows that a short line segment connecting points $(x_3, 0)$ and $(x_4, 1/2)$, after one iteration, goes to a long segment connecting points $(x_3, 0)$ and $(x_4, \mu/2)$. From these, we conclude that the action of this map on line segments of these two types is described as:

$$\begin{aligned} \mathrm{B}_\mu : \mathrm{L} &\to \mathrm{LS} \\ \mathrm{S} &\to \mathrm{L} \end{aligned} \tag{11}$$

B_μ is then also a discrete map for L and S. From (11), Proposition 3.2 follows immediately.

Proposition 3.2: The graph of $y = \mathrm{B}_\mu^{[n]}(x)$ in the range $0 \le x \le 1$ contains line segments of only two types, type L and type S, and the number of these line segments is$2F_{n+1}$.

Figure 6 shows the graph of $\mathrm{B}_\mu^{[4]}$. We can count the number of line segments in this graph to be $2F_5 = 10$.

Proposition 3.3: A simple identity, $\mu F_n + F_{n-1} = \mu^n$.

Proof: The proof is similar to the triangular T_μ map with $\mu = (1+\sqrt{5})/2$ in Proposition 2.3.

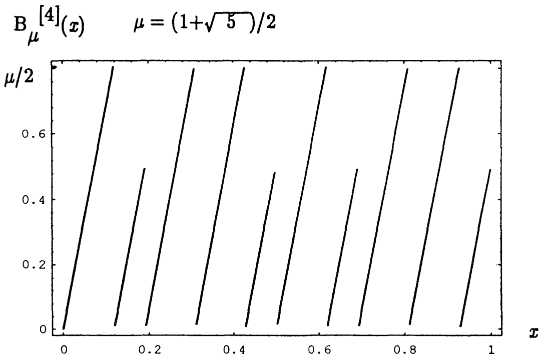

FIGURE 6

Proposition 3.4: There is an infinite LS sequence which is invariant under the action of map (11).

Proof: We denote by LS[B^n] the LS sequence describing the shape of the graph $y = B_\mu^{[n]}(x)$ in the range $0 \le x \le 1/2$. From (11), we have the following results:

$$\begin{aligned}
&\mathrm{LS}[\mathrm{B}^1] = \mathrm{L} && \\
&\mathrm{LS}[\mathrm{B}^2] = \mathrm{LS} && \\
&\mathrm{LS}[\mathrm{B}^3] = \mathrm{LSL} && = \mathrm{LS}[\mathrm{B}^2]\mathrm{LS}[\mathrm{B}^1] \\
&\mathrm{LS}[\mathrm{B}^4] = \mathrm{LSLLS} && = \mathrm{LS}[\mathrm{B}^3]\mathrm{LS}[\mathrm{B}^2] \\
&\mathrm{LS}[\mathrm{B}^5] = \mathrm{LSLLSLSL} && = \mathrm{LS}[\mathrm{B}^4]\mathrm{LS}[\mathrm{B}^3] \\
&\dots && \\
&\mathrm{LS}[\mathrm{B}^n] = \mathrm{LS}[\mathrm{B}^{n-1}]\mathrm{LS}[\mathrm{B}^{n-2}] &&
\end{aligned} \tag{12}$$

The length of the LS[B^n] sequence is F_{n+1}. Equation (12) shows that, after one iteration, an LS sequence gets longer, but the original LS sequence remains. By taking $n \to \infty$, we then have an infinite LS sequence which is invariant under the action of map (11) since, from (12), we have

$$\lim_{n\to\infty} \mathrm{LS}[\mathrm{B}^n] = \lim_{n\to\infty} \mathrm{LS}[\mathrm{B}^{n-1}].$$

We denote by $\{B^\infty\}$ the infinite LS sequence obtained from iterates of an L, that is,

$$\{\mathrm{B}^\infty\} = \lim_{n\to\infty} \mathrm{B}_\mu^{[n]}(\mathrm{L}).$$

We see that $\{B^\infty\}$ is invariant under the action of map (11). It is impossible to have an invariant sequence with S as the first element. Therefore, we have only one infinite LS sequence that is invariant under the action of map (11). Therefore, from the first few iterates of L, we have the following first few elements of this infinite sequence:

$$\{\mathrm{B}^\infty\} = \mathrm{LSLLSLSLLSLLSLSLLSLSL} \dots \tag{13}$$

Thus, to have an invariant LS sequence, the L's and S's should be arranged according to the order described in (13). We shall discuss this special symbol sequence in more detail. Denoting by $B(k)$ the k^{th} element of this sequence, then, from (13), we have $B(k) = L$ for k = 1, 3, 4, 6, 8, 9, 11, 12, 14, ..., and $B(k) = S$ for k = 2, 5, 7, 10, 13, 15, ... It would be interesting to find a way of determining that $\mathrm{B}(k)$ is an L or an S for a given k. So far, we have not obtained a simple formula for this, except for the following descriptions that are based on the following theorem.

Theorem: For $k \geq 3$, $\mathrm{B}(k) = \mathrm{B}(d)$, where $k = d + F_{n(k)}$ and $F_{n(k)}$ is the largest Fibonacci number that is less than k.

Proof: Suppose $F_{n(k)}$ is the largest Fibonacci number that is less than k, and let $d = k - F_{n(k)}$. Then $F_{n(k)} < k \leq F_{n(k)+1}$. Using the property that the length of the LS[B^n] sequence is F_{n+1}, this k^{th} element is then in the LS[$\mathrm{B}^{n(k)}$] sequence. Since $F_{n(k)} < k$, and using also the property that

$$\mathrm{LS}[\mathrm{B}^n] = \mathrm{LS}[\mathrm{B}^{n-1}]\mathrm{LS}[\mathrm{B}^{n-2}] = (\mathrm{LS}[\mathrm{B}^{n-2}]\mathrm{LS}[\mathrm{B}^{n-3}])\,\mathrm{LS}[\mathrm{B}^{n-2}],$$

we find that the k^{th} element in the LS[$\mathrm{B}^{n(k)}$] sequence is equivalent to the $(k - F_{n(k)})^{\text{th}}$ element in the LS[$\mathrm{B}^{n(k)-2}$] sequence. We then have the result that $\mathrm{B}(k) = \mathrm{B}(k - F_{n(k)}) = \mathrm{B}(d)$.

The determination of $\mathrm{B}(k)$ is then reduced to the determination of $\mathrm{B}(d)$; we call this the **reduction rule**. According to this reduction rule, we may reduce the original number k down to a final number d_k with $d_k = 1$ or 2, that is,

$$k - \sum_{n=3}^{n(k)} c_n F_n = d_k, \text{ where } c_n = 0 \text{ or } 1, \text{ and } d_k = 1 \text{ or } 2. \tag{14}$$

Equation (14) means that, for a given number $k \geq 3$, we subtract successively appropriate different Fibonacci numbers from k, until the final reduced number, d_k, is one of the two values $\{1, 2\}$. $\mathrm{B}(k)$ is then the same as $\mathrm{B}(d_k)$. We call d_k the residue of the number k. We conclude that

$$\mathrm{B}(k) = \mathrm{B}(d_k). \tag{15}$$

Since $\mathrm{B}(1) = \mathrm{L}$ and $\mathrm{B}(2) = \mathrm{S}$, we have

$$\begin{cases} \mathrm{B}(k) = \mathrm{L}, & \text{if } d_k = 1, \\ \mathrm{B}(k) = \mathrm{S}, & \text{if } d_k = 2. \end{cases} \tag{16}$$

For example, if we are to determine B(27), then as $27 = F_8 + 6$ and $6 = F_5 + 1$, we have B(27) = B(6) = B(1) = L. The method of reducing a number k down to d_k mentioned above is unique. We note that if $\mathrm{B}(k)$ = S, then $\mathrm{B}(k-1) = \mathrm{B}(k+1) = \mathrm{L}$, since if $d_k = 2$, then the coefficient c_3 in (14) must be zero; otherwise, we would have $d = 4$, and then $d_k = 1$ from the reduction rule. This contradicts $d_k = 2$; hence, $k = \sum_{n=4}^{n(k)} c_n F_n + 2$. Therefore,

$$k - 1 = \sum_{n=4}^{n(k)} c_n F_n + 1 \quad \text{and} \quad k + 1 = \sum_{n=4}^{n(k)} c_n F_n + 3 = \sum_{n=3}^{n(k)} c_n F_n + 1;$$

hence, $\mathrm{B}(k-1) = \mathrm{B}(1) = \mathrm{L}$ and $\mathrm{B}(k+1) = \mathrm{B}(1) = \mathrm{L}$. As a result, the two neighbors of an S in the sequence must be L's, or there are no two successive elements that are both S's. On the other hand, if $\mathrm{B}(k) = \mathrm{L}$, it is possible that $\mathrm{B}(k+1) = \mathrm{L}$. This occurs when k can be reduced to 3, that is $k = \sum_{n=5}^{n(k)} c_n F_n + 3$, we see that $\mathrm{B}(k) = \mathrm{B}(3) = \mathrm{L}$ and $\mathrm{B}(k+1) = \mathrm{B}(4) = \mathrm{L}$. Now, since $k + 2 = \sum_{n=3}^{n(k)} c_n F_n + 2$, we have $\mathrm{B}(k+2)$ = B(2) = S; therefore, it is impossible to have three successive elements that are all L's. We present the following table:

d_k	B(k)	k														
($d_k=1$)	L|	1	3	4	6	8	9	11	12	14	16	17	19	21	22	...
($d_k=2$)	S|	2		5	7		10		13	15		18	20		23	...

Since the two neighbors of an S are L's, it is better that we use the first three elements as the set of residues. That is, we start from B(1) = L, B(2) = S, B(3) = L, and then we have

$$\begin{cases} \mathrm{B}(k)=\mathrm{L}, & \text{if } d_k = 1 \text{ or } 3, \\ \mathrm{B}(k)=\mathrm{S}, & \text{if } d_k = 2. \end{cases} \tag{17}$$

For the case $\mathrm{B}(k)=\mathrm{S}$, we would expect that $\mathrm{B}(k-1)=\mathrm{L}$, with $d_{k-1}=1$, and $\mathrm{B}(k+1)=\mathrm{L}$, with $d_{k+1}=3$; however, this is not so. For example, when $k=5$, $\mathrm{B}(5)=\mathrm{S}$, and according to the reduction rule, we have $4=F_4+1$, so B(4) = L and $d_4=1$; but $6=F_5+1$, so B(6) = L and $d_6=1$ not 3. Instead, if we now write $6=F_4+3$ and assign $d_6=3$, then we would obtain the same result: B(6) = B(3) = L. This shows that we may assign d_6 as either 1 or 3. Numbers whose residue can be assigned as either 1 or 3 are

$$k=\sum_{n=6} c_nF_n+6, \quad c_n = 0 \text{ or } 1. \tag{18}$$

This enables us to present the following table:

d_k	B(k)	k												
($d_k=1$)	L|	1	4	6	9	12	14	17	19	22	25	27	30	...
($d_k=2$)	S|	2	5	7	10	13	15	18	20	23	26	28	31	...
($d_k=3$)	L|	3	6	8	11	14	16	19	21	24	27	29	32	...

The first, second, and third lines are, respectively, for those k with $d_k=1, 2$, and 3. We note that numbers, such as 6, 14, 19, 27, ... etc., appear both in the first and third lines. From (17), we see that there are two cases, $d_k=1$ or $d_k=3$, for $\mathrm{B}(k)$ being an L, and only one case, $d_k=2$, for $\mathrm{B}(k)$ being an S. Therefore, naively, we would think that the ratio of the total number of k with $\mathrm{B}(k)=\mathrm{L}$ to that with $\mathrm{B}(k)=\mathrm{S}$ is 2; however, this is due to the fact that we have made the double counting on those numbers of the form (18). Without the double counting, the ratio should be less than 2; indeed, the actual ratio is the golden mean $\approx 1.618<2$. It can be seen from (12) that the length of the LS[n] sequence is F_{n+1}, and that among the elements in this sequence there are F_n L's and F_{n-1} S's. Therefore, the asymptotic ratio is

$$\lim_{n\to\infty}(F_n/F_{n-1})=(1+\sqrt{5})/2.$$

We also have the following results:

(1) If $\mathrm{B}(k)=\mathrm{S}$, then $\mathrm{B}(k-1)=\mathrm{B}(k+1)=\mathrm{L}$.

(2) If $\mathrm{B}(k)=\mathrm{L}$ and k is of the form (18), then $\mathrm{B}(k+1)=\mathrm{B}(k-1)=\mathrm{S}$.

(3) If $\mathrm{B}(k)=\mathrm{L}$ and k is not of the form (18), then $\begin{cases} \text{for } d_k=1,\ \mathrm{B}(k+1)=\mathrm{S} \text{ and } \mathrm{B}(k-1)=\mathrm{L}, \\ \text{for } d_k=3,\ \mathrm{B}(k+1)=\mathrm{L} \text{ and } \mathrm{B}(k-1)=\mathrm{S}. \end{cases}$

4. A FINAL REMARK

In a triangular map, requiring that $x = 0$ be a period-3 point determines $\mu = (1+\sqrt{5})/2 \equiv \Sigma_3$, and we find the Fibonacci numbers in this map. This is not an accident, since Σ_3 is not an arbitrary number but, in fact, equals $\lim_{n\to\infty}(F_n / F_{n-1})$, the limit ratio of the Fibonacci sequence. We can generalize the above results by requiring that $x = 0$ be a period-m point [3]. This then determines a unique parameter value $\mu = \Sigma_m$ in the range $\Sigma_{m-1} < \mu < 2$. Correspondingly, the numbers of line segments in a triangular map with $\mu = \Sigma_m$ are those of the Fibonacci numbers of degree m. These are not accidents either, since Σ_m in fact equals $\lim_{n\to\infty} F_n^{(m)} / F_{n-1}^{(m)}$, the limit ratio of the corresponding generalized Fibonacci sequence, where $F_n^{(m)}$ is the n^{th} Fibonacci number of degree m. The same applies also to Baker's map with $\mu = \Sigma_m$. Therefore, following the same methods used in this paper, one would also obtain a simple identity and some invariant sequences from these two maps with $\mu = \Sigma_m$.

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