# ON THE EXISTENCE OF COUPLES OF SECOND-ORDER LINEAR RECURRENCES WITH RECIPROCAL REPRESENTATION PROPERTIES FOR THEIR FIBONACCI SEQUENCES

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The aim of this note is to show that for any given second-order linear recurrence on the complex field

$$r_{n+2} - ar_{n+1} + br_n = 0, \tag{R1}$$

where $\Delta = a^2 - 4b \neq 0$ and $b \neq 0$, another one exists such that it is possible to represent the generalized Fibonacci numbers of any of them with sums of the generalized Fibonacci numbers of the other one, with a set of coefficients to be detailed later.

To establish this property, we need the following lemmas.

***Lemma 1:*** Let $U_n(a, b)$ denote the $n^{\text{th}}$ generalized Fibonacci number of the (R1) recursion. That is, $U_{n+2} - aU_{n+1} + bU_n = 0$, $U_0 = 0$, $U_1 = 1$, where $\Delta = a^2 - 4b \neq 0$ and $b \neq 0$. Let $\sqrt{b}$ denote any of the roots of the equation $z^2 = b$. Then

$$U_{n+1}(a, b) = \sum_{p=0}^{n} \binom{2n+1-p}{p} (a - 2\sqrt{b})^{n-p} (\sqrt{b})^p. \tag{F1}$$

***Lemma 2:*** If $S$ is the set of all the couples of complex numbers $(u, v)$, their order being indifferent [that is, $(u, v) = (v, u)$], and if $T$ is the transformation defined on all $S$ by

$$T(u, v) = \left( \frac{u+v}{2} + \sqrt{uv},\ \frac{u+v}{2} - \sqrt{uv} \right),$$

then $T^2(u, v) = (u, v)$, where $T^2$ is the second iterate of $T$.

***Proof of Lemma 1:*** Edouard Lucas [1] proved that if $U_n(t, s)$ is the $n^{\text{th}}$ generalized Fibonacci number of the recursion defined on the complex field by $r_{n+2} - tr_{n+1} + sr_n = 0$, then

$$U_{n+1}(t, s) = \sum_{p=0}^{[n/2]} \binom{n-p}{p} (t)^{n-2p} (-s)^p.$$

Throughout the rest of this paper, when we refer to the *characteristic roots* of a linear recursion we mean the roots of its auxiliary algebraic equation.

Now let $\alpha$ and $\beta$ be the characteristic roots (supposed *distinct*) of the recursion (R1), let $\sqrt{\alpha}$ be any root of the equation $z^2 = \alpha$, and let $-\sqrt{\beta}$ be any root of the equation $z^2 = \beta$.

If $Y_n$ is the $n^{\text{th}}$ generalized Fibonacci number of the second-order linear recursion whose characteristic roots are $\sqrt{\alpha}$ and $-\sqrt{\beta}$ then, using Lucas' formula, we obtain

$$Y_{n+1} = \sum_{p=0}^{[n/2]} \binom{n-p}{p} (\sqrt{\alpha} - \sqrt{\beta})^{n-2p} (\sqrt{\alpha}\sqrt{\beta})^p.$$

Now, using the usual Binet form, we easily obtain

$$Y_{2(n+1)} = (\sqrt{\alpha} - \sqrt{\beta})U_{n+1}(a, b),$$

whence

$$U_{n+1}(a, b) = \sum_{p=0}^{n} \binom{2n+1-p}{p} (\sqrt{\alpha} - \sqrt{\beta})^{2(n-p)} (\sqrt{\alpha}\sqrt{\beta})^p .$$

But we have $(\sqrt{\alpha} - \sqrt{\beta})^2 = \alpha + \beta - 2\sqrt{\alpha}\sqrt{\beta} = a - 2\sqrt{\alpha}\sqrt{\beta}$.

Since $\alpha\beta = b$, it is obvious from the above definitions of $\sqrt{\alpha}$ and $\sqrt{\beta}$ that we can replace $\sqrt{\alpha}\sqrt{\beta}$ by any of the roots of $z^2 = b$. This completes the demonstration.

Since $\sqrt{b}$ may be *any* of the roots of $z^2 = b$, the following formula is also true:

$$U_{n+1}(a, b) = \sum_{p=0}^{n} \binom{2n+1-p}{p} (a + 2\sqrt{b})^{n-p} (-\sqrt{b})^p . \tag{F2}$$

***Proof of Lemma 2:*** The proof is immediate by directly computing

$$T\left(\frac{u+v}{2} + \sqrt{uv},\ \frac{u+v}{2} - \sqrt{uv}\right).$$

Now, to the recursion (R1), let us associate the recursion (R2), whose characteristic roots are $a/2 + \sqrt{b}$ and $a/2 - \sqrt{b}$, that is, the one defined by:

$$r_{n+2} - ar_{n+1} + (\Delta/4)r_n = 0, \tag{R2}$$

where $\Delta$ is the discriminant of (R1).

It is immediate that the couple of roots of (R2) are obtained by applying the $T$ transformation to the couple of roots of (R1). Therefore, by applying the same transformation to the couple of roots of (R2), we obtain the couple of roots of (R1), according to Lemma 2. Then the associate recursion for (R2) is (R1).

Now we may write (F1) and (F2) as follows:

$$U_{n+1}(a, b) = U_{n+1} = \sum_{p=0}^{n} \binom{2n+1-p}{p} 2^{n-p} (a/2 - \sqrt{b})^{n-p} (\sqrt{b})^p ,$$

$$U_{n+1}(a, b) = U_{n+1} = \sum_{p=0}^{n} \binom{2n+1-p}{p} 2^{n-p} (a/2 + \sqrt{b})^{n-p} (-\sqrt{b})^p .$$

Letting $(\Phi_n)$ be the generalized Fibonacci sequence of (R2), we may write the following formulas which are easily obtained by induction:

$$(a/2 - \sqrt{b})^{n-p} = \Phi_{n-p+1} - \Phi_{n-p}(a/2 + \sqrt{b}),$$

$$(a/2 + \sqrt{b})^{n-p} = \Phi_{n-p+1} - \Phi_{n-p}(a/2 - \sqrt{b}).$$

By substitutions in the previous formulas, we obtain

$$U_{n+1} = \sum_{p=0}^{n} \binom{2n+1-p}{p} 2^{n-p} (\sqrt{b})^p (\Phi_{n-p+1} - \Phi_{n-p}(a/2 + \sqrt{b})),$$

$$U_{n+1}=\sum_{p=0}^{n}\binom{2n+1-p}{p}2^{n-p}(-\sqrt{b})^{p}(\Phi_{n-p+1}-\Phi_{n-p}(a/2-\sqrt{b})),$$

and, summing both relations, we have

$$\begin{aligned}U_{n+1}=&\sum_{\substack{p\text{ even}\\ p\le n}}\binom{2n+1-p}{p}2^{n-p}(b)^{p/2}(\Phi_{n-p+1}-a/2\Phi_{n-p})\\ &-\sum_{\substack{p\text{ odd}\\ p\le n}}\binom{2n+1-p}{p}2^{n-p}(b)^{(p+1)/2}\Phi_{n-p}.\end{aligned}\tag{S1}$$

Since the associate recursion for (R2) is (R1), we have, symmetrically,

$$\begin{aligned}\Phi_{n+1}=&\sum_{\substack{p\text{ even}\\ p\le n}}\binom{2n+1-p}{p}2^{n-p}\left(\frac{\Delta}{4}\right)^{p/2}(U_{n-p+1}-a/2U_{n-p})\\ &-\sum_{\substack{p\text{ odd}\\ p\le n}}\binom{2n+1-p}{p}2^{n-p}\left(\frac{\Delta}{4}\right)^{(p+1)/2}U_{n-p},\end{aligned}\tag{S2}$$

because the fact that $4b\neq 0$, $4b$ being the discriminant of (R2), allows the same treatment for Lucas' formula for $\Phi_{n+1}$ as the one for $U_{n+1}$.

***Remarks:***

*1.* Do there exist recursions which are their own associates? (R1) will be so if and only if $b=\Delta/4 \Leftrightarrow b=(a^2)/8$. Therefore, a necessary and sufficient condition for a recursion to be its own associate is to assume the form

$$r_{n+2}-ar_{n+1}+(a^2)/8r_n=0,$$

where $a$ is an arbitrary nonzero complex number. Its characteristic roots are $a\sqrt{2}(\sqrt{2}+1)/4$ and $a\sqrt{2}(\sqrt{2}-1)/4$. Within the first pair of parentheses is the greatest root of the Pell recurrence, $r_{n+2}-2r_{n+1}-r_n=0$, while within the second pair is the opposite of the remaining root of the Pell recurrence. This allows us to obtain sum formulas specific for Pell and Pell-Lucas numbers, thanks to (S1).

*2.* To any second-order linear recursion, we may also associate the auxiliary polynomial of its associate recursion. That is, to the recursion defined by $r_{n+2}-ar_{n+1}+br_n=0$, associate the polynomial $x^2-ax+\Delta/4$. With this meaning, it appears that the associate polynomial for the general second-order linear recursion has been mentioned in the literature at least once, because Richard André-Jeannin [2] proved the following orthogonality property (with our notations):

$$\int_{-a-2\sqrt{b}}^{-a+2\sqrt{b}}\sqrt{x^2+2ax+\Delta}\,U_n(a+x,b)U_p(a+x,b)\,dx=0$$

for $n\neq p$, and it is obvious that the polynomial under the radical is equal to $4p\left(-\frac{x}{2}\right)$, where $p(x)$ is the associate polynomial for the recursion (R1).

With a trivial change of variable, the orthogonality relation may be written as

$$\int_h^k \sqrt{p(x)}\, U_n(a-2x, b)\, U_p(a-2x, b)\, dx = 0$$

where $h$ and $k$ are the roots of $p(x)$: $a/2-\sqrt{b}$ and $a/2+\sqrt{b}$.

## REFERENCES

1. Edouard Lucas. *The Theory of Simply Periodic Numerical Functions*. Tr. from the French by Sidney Kravitz. Santa Clara, Calif.: The Fibonacci Association, 1969.
2. Richard André-Jeannin. "A Note on a General Class of Polynomials." *The Fibonacci Quarterly* **32.5** (1994):445-64.

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