COUNTING THE NUMBER OF EQUIVALENCE CLASSES OF (m, F) SEQUENCES AND THEIR GENERALIZATIONS

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1. INTRODUCTION

K. T. Atanassov and others, in [3], [1], and [2], introduced $(2, F)$ and $(3, F)$ sequences which were pairs and triples of sequences defined by two or three simultaneous Fibonacci-like recurrences, respectively, for which the exact definition will be given at the end of this section.

There are four $(2, F)$ sequences, among which one is a pair of $(1, F)$ sequences defined by the original Fibonacci recurrence and the other three are essential. As we are interested in the solutions of the systems of recurrence equations with the general initial conditions rather than the resulting sequences for some particular initial conditions, we call such a system a "$(2, F)$ system." The $(2, F)$ system consisting of two $(1, F)$ recurrences is called a "separable $(2, F)$ system," and the other three are called "inseparable $(2, F)$ systems."

In the case of three sequences, some of the thirty-six $(3, F)$ systems of simultaneous recurrence equations give the same triple of sequences apart from their order provided appropriate initial conditions. K. T. Atanassov [2] and W. R. Spickerman et al. [5] studied equivalence classes of $(3, F)$ systems of recurrences which give essentially the same sequences and determined eleven classes. One of them consists of three $(1, F)$ recurrence equations and three of them are separated into one $(1, F)$ recurrence and an inseparable $(2, F)$ system of recurrence equations. Therefore, we have seven classes of inseparable $(3, F)$ systems of recurrence equations, for which the definition will be given in Section 4.

The purposes of this paper are to establish the method of counting the number of equivalence classes of (m, F) systems consisting of m Fibonacci-like recurrences and the number of classes of inseparable (m, F) systems, and give their values for small m. Furthermore, we apply the same method to $(m, F^{(f)})$ systems where the Fibonacci-like recurrences in (m, F) systems are replaced with f^{th}-order recurrences of type (1). More precisely, an $(m, F^{(f)})$ system is defined as follows.

Definition 1: A set of m recurrence equations

$$F_{n+1}^{(f)}(k) = F_n^{(f)}(\sigma_1(k)) + F_{n-1}^{(f)}(\sigma_2(k)) + \cdots + F_{n-f+1}^{(f)}(\sigma_f(k)) \quad (\text{for } n \geq f), \tag{1}$$

where $k = 1, 2, \ldots, m$ and $\sigma_1, \sigma_2, \ldots, \sigma_f$ are permutations belonging to the symmetric group S_m of order m is called an $(m, F^{(f)})$ system, and a set of m sequences $\{F_n^{(f)}(k)\}$, where $k = 1, 2, \ldots, m$ and $n = 1, 2, \ldots, \infty$, or a sequence of m-dimensional vectors that can be determined as the solutions of this system with given initial values $\{F_n^{(f)}(k)\}$, where $k = 1, 2, \ldots, m$ and $n = 1, 2, \ldots, f$, is called an $(m, F^{(f)})$ sequence. In particular, in the case $f = 2$, it is called an (m, F) sequence.

2. PREPARATION FROM GROUP THEORY

First, we recall a counting theorem given by Burnside.

Burnside's Theorem: Let G be a finite group of order $|G|$ operating on a finite set M. Then the number of distinct orbits associated with G is given by

$$\frac{1}{|G|}\sum_{g\in G}\lambda_1(g),$$

where $\lambda_1(g)$ is the number of fixed points in M by g.

The proof can be found, for instance, in [4] and will be omitted here.

Now, let p_m denote the number of conjugate classes in S_m, and let $b_i = |B_i|$ be the number of elements of the conjugate class B_i for $i = 1, 2, \ldots, p_m$. Each $\sigma \in S_m$ can be represented as the product of disjoint cycles uniquely up to their order. If σ is represented as the product of λ_1 cycles of length 1, λ_2 cycles of length 2, ..., λ_m cycles of length m, we say that it has the cycle type

$$1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m}, \tag{2}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are nonnegative integers satisfying

$$1\cdot\lambda_1 + 2\cdot\lambda_2 + \cdots + m\cdot\lambda_m = m. \tag{3}$$

Two permutations in S_m are conjugate if and only if they have the same cycle type since an element $\eta \in S_m$ satisfies $\eta\sigma\eta^{-1} = \sigma$ if and only if it does not change each cycle of σ or just make some permutations of the cycles of the same length. Since this gives also the condition that $\eta \in S_m$ satisfies $\eta\sigma = \sigma\eta$, the centralizers of the elements of B_i in S_m must have the same order, which will be denoted by c_i. Since all permutations in B_i have the same cycle type, we can represent it by (2). Then we have $b_i = m!/(\lambda_1!\lambda_2!\ldots\lambda_m!1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m})$ and

$$c_i = \lambda_1!\lambda_2!\ldots\lambda_m!1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m} \tag{4}$$

so that the relation

$$b_i c_i = |S_m| = m! \quad (\text{for } i = 1, 2, \ldots, p_m) \tag{5}$$

always holds.

The conjugate classes of cycle types of S_m bijectively correspond to the integer partitions of m, and an algorithm for listing them can be found, for example, in D. Stanton and D. White [6].

3. THE EQUIVALENCE CLASSES OF $(m, F^{(f)})$ SYSTEMS

First, we consider $(m, F^{(f)})$ systems. Following the manner that K. T. Atanassov did for $m = 2$ and 3, for each $m > 0$, an $(m, F^{(f)})$ system is defined by m simultaneous recurrence equations $F_{n+1}(k) = F_n(\sigma_1(k)) + F_{n-1}(\sigma_2(k))$, for $n \geq 3$, where $k = 1, 2, \ldots, m$ and σ_1 and σ_2 are permutations in S_m. This is the special case of $(m, F^{(f)})$ systems of recurrence equations defined by (1) for $f = 2$. If we give any initial values $F_n(k)$, where $k = 1, 2, \ldots, m$ and $n = 1, 2$, then an (m, F) sequence $\{F_n(k)\}$, where $k = 1, 2, \ldots, m$ and $n = 1, 2, \ldots, \infty$, will be determined by these recurrences. Since this (m, F) system is determined depending only on σ_1 and σ_2, it will be denoted by $S(\sigma_1, \sigma_2)$.

Definition 2: Two (m, F) systems $S(\sigma_1, \sigma_2)$ and $S(\tau_1, \tau_2)$ are said to be equivalent if there is an $\eta \in S_m$ such that $\eta\sigma_1\eta^{-1} = \tau_1$ and $\eta\sigma_2\eta^{-1} = \tau_2$ are satisfied.

It is shown in W. R. Spickerman et al. [5] that two $(3, F)$ systems are equivalent if and only if they define the same triple of sequences up to their order by choosing appropriate initial values of one of them for the given initial values of the other.

We define the operation of $\eta \in S_m$ on the system $S(\sigma_1, \sigma_2)$ by

$$\eta(S(\sigma_1, \sigma_2)) = S(\eta\sigma_1\eta^{-1}, \eta\sigma_2\eta^{-1}). \tag{6}$$

Assuming that the group acts on the set $M = \{S(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in S_m\}$ in this manner, we apply Burnside's theorem.

Let η be an element of S_m. Then η leaves $S(\sigma_1, \sigma_2)$ fixed if and only if $\eta\sigma_1\eta^{-1} = \sigma_1$ and $\eta\sigma_2\eta^{-1} = \sigma_2$, or $\eta\sigma_1 = \sigma_1\eta$ and $\eta\sigma_2 = \sigma_2\eta$. If $\eta \in B_i$, the number of such σ_1 and σ_2 are both c_i, so that c_i^2 of $S(\sigma_1, \sigma_2)$ will be fixed by η. Since we have b_i permutations in B_i, the number of systems fixed by permutations in a conjugate class B_i sums to $b_i c_i^2$. If we denote the number of distinct orbits in M associated with S_m, i.e., the number of equivalence classes in M by $N(m, F)$, using Burnside's theorem and relation (5), we can represent it as

$$N(m, F) = (\Sigma b_i c_i^2)/|S_m| = \Sigma c_i, \tag{7}$$

where the summation is taken over all the conjugate classes of S_m, and we can evaluate this value by (4).

We can easily generalize this result to the $(m, F^{(f)})$ system $S(\sigma_1, \sigma_2, \ldots, \sigma_f)$ which is defined by the recurrences (1).

Definition 3: Two $(m, F^{(f)})$ systems $S(\sigma_1, \sigma_2, \ldots, \sigma_f)$ and $S(\tau_1, \tau_2, \ldots, \tau_f)$ are said to be equivalent if there is an $\eta \in S_m$ such that $\eta\sigma_1\eta^{-1} = \tau_1$, $\eta\sigma_2\eta^{-1} = \tau_2, \ldots,$ and $\eta\sigma_f\eta^{-1} = \tau_f$ are satisfied.

Using the operation of $\eta \in S_m$ on $(m, F^{(f)})$ systems defined by

$$\eta(S(\sigma_1, \sigma_2, \ldots, \sigma_f)) = S(\eta\sigma_1\eta^{-1}, \eta\sigma_2\eta^{-1}, \ldots, \eta\sigma_f\eta^{-1}) \tag{8}$$

instead of (6), we will have the formula for the number of equivalence classes of $(m, F^{(f)})$ systems $N(m, F^{(f)})$, in a manner similar to the case of (m, F) systems as

$$N(m, F^{(f)}) = (\Sigma b_i c_i^f)/|S_m| = \Sigma c_i^{f-1}.$$

Thus, we have the following theorem.

Theorem 1: The number, $N(m, F^{(f)})$, of equivalence classes of the set of $(m, F^{(f)})$ systems $S(\sigma_1, \sigma_2, \ldots, \sigma_f)$ defined by the recurrences (1) is given by $N(m, F^{(f)}) = \Sigma c_i^{f-1}$, where $c_i = \lambda_1!\lambda_2!\ldots\lambda_m! 1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m}$, and the summation is taken over p_m congruent classes in S_m corresponding to the sets of nonnegative integers $\lambda_1, \lambda_2, \ldots, \lambda_m$ satisfying (3). In particular, for $f = 2$, we have $N(m, F) = \Sigma c_i$.

For $f = 1$, the value of $N(m, F^{(1)})$ represents the number p_m of congruent classes in S_m, which is also the number of integer partitions of m. This number can be calculated by any algorithm for finding all the cycle types in S_m.

If $p_k(r)$ denotes the number of integer partitions of k into exactly r parts, we can also calculate the value of p_m directly using the following properties:

(i) For $k > 0$, $p_k(1) = p_k(k) = 1$, and $p_k(r) = 0$ if $r > k$.

(ii) If $k > r > 0$, $p_k(r) = p_{k-r}(1) + p_{k-r}(2) + \cdots + p_{k-r}(r)$.

(iii) $p_m = p_m(1) + p_m(2) + \cdots + p_m(m)$.

The values of $N(m, F^{(f)})$ for small m and f are shown in Table 1.

TABLE 1

f \ m	1	2	3	4	5	6	7
1	1	2	3	5	7	11	15
2	1	4	11	43	161	901	5579
3	1	8	49	681	14721	524137	25471105
4	1	16	251	14491	1730861	373486525	128038522439

4. THE NUMBER OF INSEPARABLE EQUIVALENCE CLASSES

As we have stated for the case $m = 2, 3$ and $f = 2$, some of the $(m, F^{(f)})$ systems can be separated into smaller systems.

Definition 4: An $(m, F^{(f)})$ system $S = S(\sigma_1, \sigma_2, \ldots, \sigma_f)$ is separable if there exists a nonempty proper subset M' of $M = \{1, 2, .., m\}$ such that M' is stable (mapped into itself) by the permutations $\sigma_1, \sigma_2, \ldots, \sigma_f$. Then the system (1) can be partitioned into an $(m', F^{(f)})$ system and an $(m'', F^{(f)})$ system corresponding to M' and its relative complement $M'' = M - M'$, where $|M'| = m'$ and $|M''| = m''$, and S is separated into an $(m', F^{(f)})$ system $S'(\sigma'_1, \sigma'_2, \ldots, \sigma'_f)$ and an $(m'', F^{(f)})$ system $S''(\sigma''_1, \sigma''_2, \ldots, \sigma''_f)$, where σ'_s and σ''_s are restrictions of σ_s on M' and M'', respectively, for $s = 1, 2, \ldots, f$. Otherwise, S is said to be inseparable.

Definition 5: An $(m, F^{(f)})$ system S is said to have type $T = 1^{\lambda_1}2^{\lambda_2} \ldots m^{\lambda_m}$, if it can be divided into $\lambda_1(1, F^{(f)})$ systems, $\lambda_2(2, F^{(f)})$ systems, ..., and $\lambda_m(m, F^{(f)})$ systems that are inseparable, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are nonnegative integers satisfying (3). If $\lambda_t > 0$, S has a subsystem of type t^{λ_t} consisting of λ_t inseparable $(t, F^{(f)})$ systems, which is referred to as the t-part of S. If $\lambda_t = 0$, we say that the t-part of S is empty.

Besides the symbol $N(m, F^{(f)})$ defined above, we need the following notations.

Notations

$S(m, F^{(f)})$: The number of equivalence classes of separable $(m, F^{(f)})$ systems.

$I(m, F^{(f)})$: The number of equivalence classes of inseparable $(m, F^{(f)})$ systems.

$N[T, F^{(f)}]$: The number of equivalence classes of $(m, F^{(f)})$ systems of type T.

When we discuss a fixed f, we sometimes abbreviate the above symbols as $N(m)$, $S(m)$, $I(m)$, and $N[T]$, omitting $F^{(f)}$.

$H(n, r)$: The number of r-combinations with repetition of n distinct things, which is given by

$$H(n, r) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{(n-1)!r!},$$

where we use the convention $H(n, 0) = 1$ for $n > 0$ as usual.

Using the notations defined above, we can state the next theorem.

Theorem 2: The numbers $N[1^{\lambda_1}2^{\lambda_2}\dots m^{\lambda_m}]$, $S(m)$, and $I(m)$ are given by the following formulas:

$$N[1^{\lambda_1}2^{\lambda_2}\dots m^{\lambda_m}] = \Pi H(I(t), \lambda_t), \tag{9}$$

where the product is taken over $t = 1, 2, \dots, m$;

$$S(m) = \Sigma N[1^{\lambda_1}2^{\lambda_2}\dots (m-1)^{\lambda_{m-1}}] \quad \text{and} \quad I(m) = N[m^1] = N(m) - S(m),$$

where the summation is taken over all the integer partitions of m into more than one part or all of the $(m-1)$-tuples of nonnegative integers $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ satisfying

$$1\cdot\lambda_1 + 2\cdot\lambda_2 + \cdots + (m-1)\lambda_{m-1} = m.$$

Proof: Let $S = S(\sigma_1, \sigma_2, \dots, \sigma_f)$ be an $(m, F^{(f)})$ system defined by (1). A system $\eta S(\sigma_1, \sigma_2, \dots, \sigma_f)$ equivalent to S, which is defined by (8), is given by replacing functions $F_s^{(f)}(x)$ in all terms of (1) with $F_s^{(f)}(\eta(x))$ for $s = n+1, n, n-1, \dots, n-f+1$, and rearranging the m equations so that $\eta(k)$'s of the left-hand side become increasing in order. If the $(m, F^{(f)})$ system S is separable, then the nonempty subsets M' and M'' in Definition 4, which are stable by $\sigma_1, \sigma_2, \dots, \sigma_f$, are mapped onto $\eta(M')$ and $\eta(M'')$, which are complements of each other and are stable by $\eta\sigma_1\eta^{-1}, \eta\sigma_2\eta^{-1}, \dots, \eta\sigma_f\eta^{-1}$. Therefore, it is clear that two equivalent systems have the same type and two systems of the same type are equivalent if and only if their t-parts are equivalent for $t = 1, 2, \dots, m$.

The equivalence class of the t-part of S will be determined by the classes of $I(t)$ to which $\lambda_t(t, F^{(f)})$ subsystems of S belong, not depending on the location or the variables used in them. Therefore, the number of equivalence classes of the t-part with type t^{λ_t} of $(m, F^{(f)})$ systems is the number of λ_t-combinations with repetition taken from $I(t)$, which is denoted by $H(I(t), \lambda_t)$. Since different choices of an equivalence class for any t-part give different equivalence classes of $(m, F^{(f)})$ systems of type $T = 1^{\lambda_1}2^{\lambda_2}\dots m^{\lambda_m}$, their number will be represented by (9).

Since $N(m, F^{(f)})$ is the sum of expression (9) for all the solutions of equation (3), and the only solution of (3) with $\lambda_m > 0$ is given by $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 0$ and $\lambda_m = 1$, and the type of an inseparable $(m, F^{(f)})$ system is m^1, we have

$$S(m) = N(m) - I(m) = N(m) - N[m^1] = \Sigma N[1^{\lambda_1}2^{\lambda_2}\dots (m-1)^{\lambda_{m-1}}],$$

and the proof is completed.

Since we have only one equivalence class for $(1, F^{(f)})$ system, the number of equivalence classes of $(m, F^{(f)})$ systems of type $1^{\lambda_1}2^{\lambda_2}\dots(m-1)^{\lambda_{m-1}}$ for which $\lambda_1 > 0$ must be equal to the number of equivalence classes of $(m-1, F^{(f)})$ systems of type $1^{\lambda_1}2^{\lambda_2}\dots(m-1)^{\lambda_{m-1}}$, so the total number of equivalence classes with nonempty 1-parts of $(m, F^{(f)})$ systems is equal to $N(m-1)$.

Since an $(m, F^{(f)})$ system with an empty 1-part cannot have an $(m-1)$-part, we have another expression for $S(m)$ and $I(m)$ that is useful for inductive calculation.

Corollary: $S(m) = N(m-1) + \Sigma H(I(2), \lambda_2)H(I(3), \lambda_3) \ldots H(I(m-2), \lambda_{m-2})$, where summation is taken over all the nonnegative integers $\lambda_2, \lambda_3, \ldots, \lambda_{m-2}$ satisfying $2 \cdot \lambda_2 + 3 \cdot \lambda_3 + \cdots + (m-2) \cdot \lambda_{m-2} = m$ and $I(m) = N[m^1] = N(m) - S(m)$.

The numbers of equivalence classes $N[T]$ for T with small values of m and $f = 2$ and 3 are given in Table 2, where the number $I(m) = N[m^1]$ of equivalence classes of inseparable $(m, F^{(f)})$ systems can be found in the right-most column for each m.

TABLE 2

$m = 2$

f \ T	1^2	2^1
2	1	3
3	1	7

$m = 3$

f \ T	1^3	$1^1 2^1$	3^1
2	1	3	7
3	1	7	41

$m = 4$

f \ T	1^4	$1^2 2^1$	$1^1 3^1$	2^2	4^1
2	1	3	7	6	26
3	1	7	41	28	604

$m = 5$

f \ T	1^5	$1^3 2^1$	$1^2 3^1$	$1^1 2^2$	$1^1 4^1$	$2^1 3^1$	5^1
2	1	3	7	6	26	21	97
3	1	7	41	28	604	287	13753

$m = 6$

f \ T	1^6	$1^4 2^1$	$1^3 3^1$	$1^2 2^2$	$1^2 4^1$	$1^1 2^1 3^1$	$1^1 5^1$	2^3	$2^1 4^1$	3^2	6^1
2	1	3	7	6	26	21	97	10	78	28	624
3	1	7	41	28	604	287	13753	84	4228	861	504243

$m = 7$

f \ T	1^7	$1^5 2^1$	$1^4 3^1$	$1^3 2^2$	$1^3 4^1$	$1^2 2^1 3^1$	$1^2 5^1$	$1^1 2^3$	$1^1 2^1 4^1$	$1^1 3^2$	$1^1 6^1$	$2^2 3^1$	$2^1 5^1$	$3^1 4^1$	7^1
2	1	3	7	6	26	21	97	10	78	28	624	42	291	182	4163
3	1	7	41	28	604	287	13753	84	4228	861	504243	1148	96271	24764	24824785

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