INITIAL VALUES FOR HOMOGENEOUS LINEAR RECURRENCES OF SECOND ORDER

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(*Submitted April 1995*)

0. INTRODUCTION

A homogeneous linear recurrence of second order with constant coefficients is a sequence of equations

$$u_{n+2} = au_{n+1} + bu_n, \quad n \geq 0, \tag{0}$$

for fixed complex numbers $a, b \neq 0$. A solution $\{u_n\}_{n\geq 0}$ is completely determined by (0) and the two initial values u_0, u_1. C. Kimberling [1] raised the following problem: under what conditions on two nonnegative integers i, j does every complex pair u_i, u_j determine the whole recurrence sequence $\{u_n\}$ with (0)? In this article, I give two answers to this question (Theorems 1 and 2; the second corrects Theorems 2 and 6 of [1]) and apply them to the properties of the initial pairs. In Theorem 3 I discuss how they are distributed, while in Theorem 4 I discuss which initial values generate a periodic sequence.

1. A FIRST CRITERION FOR INITIAL PAIRS

Given a recurrence (0), we call two nonnegative numbers $i < j$ an "initial pair" if, for all complex numbers c_i, c_j, there exists one and only one solution $\{u_n\}$ of (0) with $u_i = c_i$, $u_j = c_j$. An initial pair is always $i, i+1$. Most pairs i, j will be initial, but there are exceptions: $0, 2$ is not an initial pair of $u_{n+2} = u_n$.

Theorem 1 ([1], Theorem 1): Given the recurrence (0) with $b \neq 0$, for every pair of nonnegative integers i, j with $i+1 < j$, the following two conditions are equivalent:

i, j is an initial pair for (0); (1)

the $(j-i-1)$-rowed matrix

$$D_{j-i} := \begin{pmatrix} a & -1 & 0 & & & \\ b & a & -1 & & & \\ 0 & b & a & & & \\ & & & \ddots & & \\ & & & b & a & -1 \\ & & & 0 & b & a \end{pmatrix} \tag{2}$$

is regular.

Proof: The pair $i, i+2$ is initial iff $a \neq 0$, since $au_{i+1} = u_{i+2} - bu_i$. So let $j > i+2$. If $u_i = c_i$ and $u_j = c_j$ are given, then the equations $bu_n + au_{n+1} - u_{n+2} = 0$, for $n = i, i+1, \ldots, j-2$, give us the system

$$
\begin{array}{lr}
au_{i+1} - \ u_{i+2} & = -bc_i \\
bu_{i+1} + au_{i+2} - \ u_{i+3} & = 0 \\
\quad bu_{i+2} + au_{i+3} - u_{i+4} & = 0 \\
\quad\quad \ddots & \vdots \\
\quad\quad bu_{j-3} + au_{j-2} - \ u_{j-1} & = 0 \\
\quad\quad\quad bu_{j-2} + au_{j-1} & = c_j .
\end{array}
$$

Now, i, j is an initial pair iff this system of $j-i-1$ linear equations has a unique solution $u_{i+1}, u_{i+2}, \ldots, u_{j-1}$ (and hence all $u_n, n \geq 0$, are determined) for all c_i, c_j. A necessary and sufficient condition for this is that the associated homogeneous linear system is only trivially soluble, hence the regularity of the coefficient matrix D_{j-i}. $\square$

Remark: This criterion can be extended to sequences of higher order (see [1], Theorem 7). Condition (1) is equivalent to the following: the monoms z^i, z^j are a basis of the complex vector-space $\mathbb{C}[z]$ of polynomials modulo the subspace $\mathbb{C}[z](z^2 - az - b)$. This was generalized by M. Peter [2] to recurrences of several variables of higher order.

2. A SECOND CRITERION FOR INITIAL PAIRS

Let $n := j - i$. We compute $d_n := \det D_n$ by expanding the determinant of D_{n+2} *à la* Laplace:

$$d_{n+2} = ad_{n+1} + bd_n, \quad d_0 := 0, \quad d_1 := 1. \tag{3}$$

Let

$$\zeta_1 := \tfrac{1}{2}(a + \sqrt{a^2 + 4b}) \quad \text{and} \quad \zeta_2 := \tfrac{1}{2}(a - \sqrt{a^2 + 4b})$$

be the zeros of the companion polynomial $z^2 - az - b$ of (0), then the solution of the initial problem (3) has the Binet representation

$$d_n = \begin{cases} \dfrac{1}{\zeta_i - \zeta_2}(\zeta_1^n - \zeta_2^n) & \text{if } \zeta_1 \neq \zeta_2, \\ n\left(\dfrac{a}{2}\right)^{n-1} & \text{if } \zeta_1 = \zeta_2, \end{cases} \tag{4}$$

for all $n \in \mathbb{N}$. Hence we get $d_n = 0 \Leftrightarrow \zeta_1 \neq \zeta_2, \quad \zeta_1^n = \zeta_2^n$. The last condition is equivalent to

$$\exists\, 1 \leq m \leq n-1: \ \zeta_1 = \exp\left(2\pi i \frac{m}{n}\right)\zeta_2.$$

We compute

$$
\begin{aligned}
\zeta_1 = \exp\left(2\pi i \frac{m}{n}\right)\zeta_2 &\Leftrightarrow a + \sqrt{a^2+4b} = \exp\left(2\pi i \frac{m}{n}\right)(a - \sqrt{a^2+4b}) \\
&\Leftrightarrow \sqrt{a^2+4b}\left(\exp\left(2\pi i \frac{m}{n}\right) + 1\right) = a\left(\exp\left(2\pi i \frac{m}{n}\right) - 1\right) \\
&\Leftrightarrow \sqrt{a^2+4b}\,\cos\left(\pi \frac{m}{n}\right) = -ia\,\sin\left(\pi \frac{m}{n}\right).
\end{aligned}
$$

This finally means

$$\exists\, 1 \le m \le n-1\colon\ a^2 = -4b\cos^2\left(\pi\frac{m}{n}\right).$$

Combining this with Theorem 1, we have

Theorem 2: Suppose we have a recurrence (0) with $b \neq 0$ and a pair of nonnegative integers $i < j$. Then the following three properties are equivalent:

i, j is an initial pair of (0); (1)

if ζ_1 and ζ_2 are the zeros of the polynomial $z^2 - az - b$, then $\zeta_1 = \zeta_2$ or $\zeta_1^{j-i} \neq \zeta_2^{j-i}$; (5)

$\frac{a^2}{4b} \neq -\cos^2\left(\pi\frac{m}{j-i}\right)$ for every $1 \le m < j-i$. (6)

Examples (cf. [1], Theorems 2-5): For each of the following cases, a necessary and sufficient condition that $i < j$ is an initial pair of (0) is

i) $a = 0$: $j - i \not\equiv 0 \bmod 2$;
ii) $a^2 = -b$: $j - i \not\equiv 0 \bmod 3$;
iii) $a^2 = -2b$: $j - i \not\equiv 0 \bmod 4$;
iv) $a^2 = -3b$: $j - i \not\equiv 0 \bmod 6$.

If $a^2 = -kb$ with $k \in \mathbb{Z} - \{0, 1, 2, 3\}$, then every pair $i < j$ is initial.

3. DISTRIBUTION OF INITIAL PAIRS IN RESIDUE CLASSES

In the examples of initial pairs $i < j$ given above, $j - i$ lies outside of some residue class. The next theorem explains why.

Theorem 3:

a) Suppose that the recurrence (0) with $b \neq 0$ has a pair that is not initial, then there exists an integer $m \ge 2$ such that, for every pair $i < j$ of nonnegative integers, we have that i, j is initial for (0) $\Leftrightarrow j - i \not\equiv 0 \bmod m$.

b) For every natural number $m \ge 2$, there is a recurrence (0) such that $0, j$ is initial for (0) $\Leftrightarrow j \not\equiv 0 \bmod m$.

Proof:

a) By Theorem 1, there exists a natural number $n \ge 2$ with $d_n = 0$. Let $m := \min\{n \ge 2\colon d_n = 0\}$ and $\delta := d_{m+1}$. From (4), we deduce that $d_{qm+r} = \delta^q d_r$ for all $q \in \mathbb{N}_0$, $0 \le r < m$. Furthermore, since $\delta \neq 0$, we have $d_n = 0 \Leftrightarrow n \equiv 0 \bmod m$.

Using Theorem 1, we see that this is equivalent to our first assertion.

b) Let $\zeta := \exp(2\pi i/m)$, $a := \zeta + 1$, $b := -\zeta$, then $d_j = (\zeta^j - 1)/(\zeta - 1)$, $j \in \mathbb{N}$, so that $d_j = 0 \Leftrightarrow j \equiv 0 \bmod m$.

Theorem 3 is proved. □

4. PERIODIC SEQUENCES

If i, j is an initial pair for (0), we now seek conditions under which two complex numbers c_i, c_j generate a *periodic* recurrence sequence $\{u_n\}$ with $u_i = c_i$ and $u_j = c_j$.

Theorem 4: Given a recurrence (0) with $b \neq 0$, a pair i, j in $\mathbb{N}_0$ with $i < j$, complex numbers c_i, c_j not both zero, and $m \in \mathbb{N}$, then the following two conditions are equivalent:

$$i, j \text{ is an initial pair for (0) and the solution } \{u_n\}_{n\geq 0} \text{ of (0) with } u_i = c_i,\ u_j = c_j \text{ has period } m. \tag{7}$$

One of these four cases is valid:

$$\left.\begin{array}{ll} \text{(a)} & \zeta_1^m = 1,\ \zeta_1^{j-i} \neq \zeta_2^{j-i},\ c_j = c_i\zeta_1^{j-i}; \\ \text{(b)} & \zeta_2^m = 1,\ \zeta_1^{j-i} \neq \zeta_2^{j-i},\ c_j = c_i\zeta_2^{j-i}; \\ \text{(c)} & \zeta_1^m = \zeta_2^m = 1,\ \zeta_1^{j-i} \neq \zeta_2^{j-i}; \\ \text{(d)} & \left(\dfrac{a}{2}\right)^2 = -b,\ \left(\dfrac{a}{2}\right)^m = 1,\ c_j = c_i\left(\dfrac{a}{2}\right)^{j-i}. \end{array}\right\} \tag{8}$$

Here again, ζ_1, ζ_2 are the zeros of $z^2 - az - b$.

Proof: Because of Theorem 2, each of the four conditions implies that i, j is an initial pair for (0). Hence, it suffices to show under which condition the unique solution $\{u_n\}$ of (0) with $u_i = c_i$ and $u_j = c_j$ has period m.

1) $\zeta_1 \neq \zeta_2$. In this case,

$$u_n = \frac{1}{\zeta_1^{j-i} - \zeta_2^{j-i}}[(c_j - c_i\zeta_2^{j-i})\zeta_1^{n-i} - (c_j - c_i\zeta_1^{j-i})\zeta_2^{n-i}], \quad n \geq 0.$$

However, the property $u_{n+m} = u_n$, $n \geq 0$, is equivalent to

$$(c_j - c_i\zeta_2^{j-i})\zeta_1^{n+m-i} - (c_j - c_i\zeta_1^{j-i})\zeta_2^{n+m-i} = (c_j - c_i\zeta_2^{j-i})\zeta_1^{n-i} - (c_j - c_i\zeta_1^{j-i})\zeta_2^{n-i},\ \forall n \in \mathbb{N}$$

$$\Leftrightarrow (c_j - c_i\zeta_2^{j-i})(\zeta_1^m - 1)\zeta_1^{n-i} = (c_j - c_i\zeta_1^{j-i})(\zeta_2^m - 1)\zeta_2^{n-i},\ \forall n \in \mathbb{N}$$

$$\Leftrightarrow \begin{cases} (c_j - c_i\zeta_2^{j-i})(\zeta_1^m - 1) = 0 \\ (c_j - c_i\zeta_1^{j-i})(\zeta_2^m - 1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{(a)} & \zeta_1^m = 1,\ c_j = c_i\zeta_1^{j-i}, \\ \text{(b)} & \zeta_2^m = 1,\ c_j = c_i\zeta_2^{j-i}, \\ \text{(c)} & \zeta_1^m = \zeta_2^m = 1, \\ \text{(d)} & c_j = c_i\zeta_1^{j-i} = c_i\zeta_2^{j-i}. \end{cases}$$

Since $\zeta_1^{j-i} \neq \zeta_2^{j-i}$, case (d) is impossible.

2) $\zeta_1 = \zeta_2$. Here $u_n = \frac{1}{j-i}\left[(n-i)c_j + (j-n)c_i\left(\frac{a}{2}\right)^{j-i}\right]\left(\frac{a}{2}\right)^{n-j}$.

One can easily compute

$$u_{n+m} = u_n,\ \forall n \geq 0 \Leftrightarrow \left(\frac{a}{2}\right)^m = 1,\ c_j = c_i\left(\frac{a}{2}\right)^{j-i},$$

which is the case (d) of (8), and Theorem 4 is proved. □

REFERENCES

1. C. Kimberling. "Sets of Terms that Determine All the Terms of a Linear Recurrence Sequence." *The Fibonacci Quarterly* **29.3** (1991):244-48.
2. M. Peter. "Rekurrente zahlentheoretische Funktionen in mehreren Variablen." (To be published.)

AMS Classification Numbers: 11A25, 11B37, 11B39

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