

# THE CONSTANT FOR FINITE DIOPHANTINE APPROXIMATION

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Let  $x$  be an irrational number. In 1891, Hurwitz [3] proved that there are infinitely many rational numbers  $p/q$  such that  $p$  and  $q$  are coprime integers and  $|x - p/q| < 1/(\sqrt{5}q^2)$ . Hurwitz' theorem has been extensively investigated (see [6]).

In 1948, following Davenport's suggestion, Prasad [4] initiated the study of finite Diophantine approximation. He proved that, for any given irrational number  $x$ , and any given positive integer  $m$ , there is a constant  $C_m$  such that the inequality  $|x - p/q| < 1/(C_m q^2)$  has at least  $m$  rational solutions  $p/q$ . In [4], the structure of  $C_m$  has been mentioned, and  $C_1 = (3 + \sqrt{5})/2$  has been calculated, but the values of  $C_m$  as a function of  $m$  is still unknown.

In this note we will use the Fibonacci sequence to prove that

$$C_m = \sqrt{5} + \frac{\sqrt{5}}{\left(\frac{7+3\sqrt{5}}{2}\right)^m - 1}. \quad (1)$$

**Theorem 1:** Let  $x$  be an irrational number. If  $m$  is a given positive integer, then there are at least  $m$  rational numbers  $p/q$  such that  $p$  and  $q$  are coprime integers and  $|x - p/q| < 1/(C_m q^2)$ , where  $C_m$  is as shown in formula (1). The constants  $C_m$  cannot be replaced by a smaller number.

**Proof:** Let  $x = [a_0; a_1, a_2, \dots, a_n, \dots]$  be the expansion of  $x$  in a simple continued fraction. Let  $p_n/q_n = [a_0; a_1, \dots, a_n]$  be the  $n^{\text{th}}$  convergent, then  $p_n$  and  $q_n$  are coprime integers. It is well known that (see [5])

$$|x - p_n/q_n| = 1/(M_n q_n^2),$$

where  $M_n = a_{n+1} + [0; a_{n+2}, a_{n+3}, \dots] + [0; a_n, a_{n-1}, \dots, a_1]$ .

By Legendre's theorem [5],  $|x - p/q| < 1/(2q^2)$  implies that  $p/q$  must be a convergent  $p_n/q_n$  for some  $n$ . Thus, we need only discuss the rational solutions of  $|x - p/q| < 1/(C_m q^2)$  among the convergents  $p_n/q_n$ .

We discuss the following possible cases on the partial quotients  $a_n$ . It is easily seen that  $C_m \leq C_1 < 8/3 < 3$ .

Suppose there are infinitely many  $a_n \geq 3$ , then  $M_{n-1} \geq a_n \geq 3 \geq C_m$  for all positive integer  $m$ . Hence, we need only consider the case in which there are only finitely many  $a_n \geq 3$ . That is to say, there is a positive integer  $N_1$  such that  $n \geq N_1$  implies  $a_n \leq 2$ . We consider two cases.

**Case 1.** There are infinitely many  $a_n$  such that  $a_n = 2$ . Then, for these  $n$ ,  $n > N_1 + 2$  implies  $M_{n-1} \geq 2 + [0; 2, 1] + [0; 2, 1] = 8/3 > C_m$  for all positive integers  $m$ .

**Case 2.** There are finitely many  $a_n = 2$ . Thus, there is a positive integer  $N_2 \geq N_1$  such that  $n \geq N_2$  implies  $a_n = 1$ .

Let  $N = \max\{n, a_n \neq 1\}$ . Then  $a_N \geq 2$ ,  $a_{N+1} = a_{N+2} = \dots = 1$ . Therefore, if we use  $[0; (1)_k]$  to denote  $[0; 1, \dots, 1]$  with  $k$  consecutive 1's, the following inequalities are true because  $a_{N+1} = a_{N+2} = \dots = a_{N+2m-1} = 1$ ; there are  $2m-1$  consecutive 1's.

$$\begin{aligned} M_{N+2m-1} &= a_{N+2m} + [0; \bar{1}] + [0; 1, \dots, 1, a_N, a_{N-1}, \dots, a_1] \\ &\geq 1 + [0; \bar{1}] + [0; (1)_{2m-1}]. \end{aligned} \quad (2)$$

Similarly, we have

$$\begin{aligned} M_{N+2m+1} &= a_{N+2m+2} + [0; \bar{1}] + [0; 1, \dots, 1, a_N, a_{N-1}, \dots, a_1] \\ &\geq 1 + [0; \bar{1}] + [0; (1)_{2m+1}], \\ &\dots, \\ M_{N+4m-3} &= a_{N+4m-2} + [0; \bar{1}] + [0; 1, \dots, 1, a_N, a_{N-1}, \dots, a_1] \\ &\geq 1 + [0; \bar{1}] + [0; (1)_{4m-3}]. \end{aligned}$$

It is easily seen that  $M_{N+2m-1} < M_{N+2m+1} < \dots < M_{N+4m-3}$ . Denoting  $C_m = M_{N+2m-1}$ , then the inequality  $|x - p/q| < 1/(C_m q^2)$  has at least  $m$  rational solutions  $p_n/q_n$ .

Now we calculate  $C_m$  with the help of the Fibonacci sequence.

Let  $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$  be the Fibonacci sequence. We are going to find a formula for  $[0; (1)_{2m-1}]$  by mathematical induction.

It is easily seen that  $[0; (1)_1] = [0; 1] = 1/1 = F_1/F_2$ . Suppose  $[0; (1)_{2k-1}] = F_{2k-1}/F_{2k}$ , then we have  $[0; (1)_{2(k+1)-1}] = [0; 1, 1, (1)_{2k-1}] = 1/(1 + (1 + F_{2k-1}/F_{2k})) = F_{2k+1}/F_{2k+2}$ . Thus,  $[0; (1)_{2m-1}] = F_{2m-1}/F_{2m}$ .

By Binet's formula for the Fibonacci sequence [1], i.e.,  $F_n = ((1 + \sqrt{5})^n - (1 - \sqrt{5})^n) / (2^n \sqrt{5})$ , we can find  $F_{2m-1}/F_{2m}$  as follows:

$$\begin{aligned} \frac{F_{2m-1}}{F_{2m}} &= \frac{2((1 + \sqrt{5})^{2m-1} - (1 - \sqrt{5})^{2m-1})}{(1 + \sqrt{5})^{2m} - (1 - \sqrt{5})^{2m}} = \frac{\sqrt{5}((1 + \sqrt{5})^{2m} + (1 - \sqrt{5})^{2m})}{2((1 + \sqrt{5})^{2m} - (1 - \sqrt{5})^{2m})} - \frac{1}{2} \\ &= \frac{\sqrt{5}(1 + (\sqrt{5} - 3)/2)^{2m}}{2(1 - (\sqrt{5} - 3)/2)^{2m}} - \frac{1}{2} = \frac{\sqrt{5}}{2} \left( 1 + \frac{2((\sqrt{5} - 3)/2)^{2m}}{1 - ((\sqrt{5} - 3)/2)^{2m}} \right) - \frac{1}{2} \\ &= \frac{\sqrt{5}}{2} \left( 1 + \frac{2}{((3 + \sqrt{5})/2)^{2m} - 1} \right) - \frac{1}{2}. \end{aligned}$$

Notice that because  $[0; \bar{1}] = (\sqrt{5} - 1)/2$  we have, by formula (2), that

$$C_m = M_{N+2m-1} = 1 + (\sqrt{5} - 1)/2 + F_{2m-1}/F_{2m},$$

which gives formula (1).

The constants  $C_m$  cannot be replaced by smaller numbers since, for  $x = [0; \bar{1}]$ , we have exactly  $C_m = M_{2m-1} = 1 + [0; \bar{1}] + [0; (1)_{2m-1}]$ .  $\square$

**Corollary 1:**  $C_1 = (3 + \sqrt{5})/2 = 2.6180,$   
 $C_2 = (7 + 3\sqrt{5})/6 = 2.2847,$   
 $C_3 = (9 + 4\sqrt{5})/8 = 2.2430.$

*Corollary 2:*  $\lim_{m \rightarrow \infty} C_m = \sqrt{5} = 2.2361.$

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