ON THE PROPORTION OF DIGITS IN REDUNDANT NUMERATION SYSTEMS*

Jon T. Butler
Dept. of Electrical and Comp. Eng., Naval Postgraduate School, Code EC/Bu, Monterey, CA 93943-5121

Tsutomu Sasao
Dept. of Computer Science and Electronics, Kyushu Institute of Technology, Iizuka 820, Japan
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1. INTRODUCTION

In the standard binary numeration system, an n-bit integer N is uniquely represented as the sum of powers of 2. Specifically,

$$N = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \cdots + a_2 2^2 + a_1 2^1 + a_0 2^0,$$

where a_i is either 0 or 1. As is common, N can be represented as an n-tuple of 0's and 1's, where the position of the bit determines the power of 2 involved. For example, in a 4-bit standard binary numeration system, $N = 0101 = 5$, since 5 is equivalent to $2^2 + 2^0$. Newman ([13], p. 2422) suggests that the Chinese used the binary numeration system around 3000 B.C.

Instead of powers of 2's, if Fibonacci numbers are used, then an alternate numeration system (viz. Zeckendorf [14]) occurs in which an integer N may have more than one representative. That is, let

$$N = a_{n-1}F_{n+1} + a_{n-2}F_n + \cdots + a_2F_4 + a_1F_3 + a_0F_2, \tag{1}$$

where F_i is the i^{th} Fibonacci number. For example, $1000 = 0110 = 5$ in the Fibonacci numeration system, where 5 is equivalent to both F_5 and $F_4 + F_3$. It is known (e.g., Brown [1]) that an n-tuple of 0's and 1's is a unique representative of N if every pair of 1's is separated by at least one 0. Under this restriction, we view 1000 as the representative of 5 and 0110 as the redundant representative. Brown [2] showed that, if one represents an integer by the n-tuple with the *most* 1's, then this representative is unique. In this case, we view 0110 as the representative of 5 and 1000 as the redundant representative.

Representations of this type have important advantages. For example, in a CD-ROM, three or more consecutive 1's cannot be read reliably (Davies [4]). Motivated by this, Klein [11] investigated Fibonacci-like representations of the form (1), where $F_i = F_{i-1} + F_{i-m}$ for $i > m+1$, and $F_i = i-1$ for $1 < i \leq m+1$. The case $m = 2$ corresponds to the Zeckendorf representation using Fibonacci numbers.

Kautz [9] uses such representations in a data transmission system where the receiver clock is synchronized to the transmitter clock using only the data. Toward this end, he uses code words in which there are neither strings of 1's of length greater than m nor strings of 0's of length greater than m.

Dimitrov and Donevsky [5] show that the number of steps required to multiply two n-bit numbers represented in the Zeckendorf numeration system using Quadranacci numbers is less than

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that required by two numbers represented as standard binary numbers. That is, even though the Zeckendorf representation requires more bits, its efficiency in the multiplication process more than compensates for extra operations because of larger word size. Indeed, the Zeckendorf representation outperforms both standard binary multiplication and multiplication using the more efficient multiplication algorithm for $n \to \infty$ when the number of bits in the standard binary representation is 131 through 1200.

The question posed and answered in this paper is: To what extent does redundancy occur in certain redundant numeration systems? The question has important consequences for both the efficiency of number representations and the transmission of data. We analyze redundancy in two ways: 1) the number of distinct representative n-tuples for some given n and 2) the proportion of digits used in nonredundant representatives. Table 1 shows the numeration systems considered in this paper and the corresponding recurrences, basis elements, and references.

TABLE 1. Selected Numeration Systems, Recurrences, and Basis Elements

Name	Recurrence	Basis elements	Reference
Standard binary	$F_i = 2F_{i-1}$	... 2^6 2^5 2^4 2^3 2^2 2^1 2^0	[7,8,12,13]
Zeckendorf - - Fibonacci	$F_i = F_{i-1} + F_{i-2}$	... 21 13 8 5 3 2 1	[1,2,3,14]
Gen. Fibonacci - - Tribonacci - - Quadranacci	$F_i = F_{i-1} + F_{i-2} + \ldots + F_{i-m}$ $F_i = F_{i-1} + F_{i-2} + F_{i-3}$ $F_i = F_{i-1} + F_{i-2} + F_{i-3} + F_{i-4}$	 ... 44 24 13 7 4 2 1 ... 56 29 15 8 4 2 1	[3,9]
Generalization of Fibonacci Numbers	$F_i = F_{i-1} + F_{i-m}$ $F_i = F_{i-1} + F_{i-3}$ $F_i = F_{i-1} + F_{i-4}$	 ... 13 9 6 4 3 2 1 ... 10 7 5 4 3 2 1	[11]
m - ary Numbers	$F_i = mF_{i-1} - F_{i-2}$ $F_i = 3F_{i-1} - F_{i-2}$ $F_i = 4F_{i-1} - F_{i-2}$	 ... 144 55 21 8 3 1 ... 780 209 56 15 4 1	[11]

2. BINARY NUMERATION SYSTEMS

Consider a numeration system in which the basis elements are $(\ldots, F_4, F_3, F_2)$, where $F_i = F_{i-1} + F_{i-2} + \cdots + F_{i-m}$ for $i > m+1$, and $F_i = 2^{i-2}$ for $1 < i \le m+1$, where $m \ge 2$. Consider a representative n-tuple $T = (a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$, where $a_i \in \{0, 1\}$. From [3] and [6], if no more than $m-1$ consecutive a_i's are 1, then T is a unique representative of $N = \sum_{i=0}^{n-1} a_i F_{i+2}$. We can write the regular expression (see [10], pp. 617-23) for the allowed representatives as

$$\mathbf{R} = (\lambda + \mathbf{1} + \mathbf{1}^2 + \mathbf{1}^3 + \cdots + \mathbf{1}^{m-1})(\mathbf{0}(\lambda + \mathbf{1} + \mathbf{1}^2 + \mathbf{1}^3 + \cdots + \mathbf{1}^{m-1}))^*. \tag{2}$$

Here, $\mathbf{a}^* = \{\lambda, a, aa, aaa, \ldots\}$, where λ is the *empty* string, and $\mathbf{1}^i$ denotes i consecutive 1's. Thus, this expression represents the set of strings consisting of substrings beginning with i 1's, for $0 \le i \le m-1$, followed by a sequence of substrings each of the form 0, 01, 011, ..., or 01^{m-1}. From (2), we can derive a generating function $N(x, y, z)$ for the number of representatives and the number of 0's and 1's in these representatives. Let x track the number of bits, y track the number of 0's, and z track the number of 1's. Then, a typical term in the power series expansion of $N(x, y, z)$ is $\xi_{nij} x^n y^i z^j$ for $n = i + j$, where ξ_{nij} is the number of representative n-tuples with i 0's and j 1's. We can write

$$N(x,y,z) = (1+xz+x^2z^2+\cdots+x^{m-1}z^{m-1})\left(\frac{1}{1-xy(1+xz+x^2z^2+\cdots+x^{m-1}z^{m-1}}\right), \tag{3}$$

where the first term represents the leftmost substring, which can be nothing, 1, 1^2, ..., or 1^{m-1}, while the second term represents the ways to choose 0, 01, 01^2, ..., and 01^{m-1}. We can rewrite (3) as follows:

$$N(x,y,z) = \left(\frac{1-x^m z^m}{1-xz}\right)\left(\frac{1}{1-xy\left(\frac{1-x^m z^m}{1-xz}\right)}\right).$$

From this we can generate, for example, the distribution of 16-tuples with i 1's for $0 \le i \le 15$, as shown in Figure 1. It is interesting that the number of representative n-tuples increases markedly from $m = 2$ to $m = 3$; for $m = 7$, the distribution is almost binomial. The fact that it is not exactly binomial can be seen by its asymmetry. Capocelli, Cerbone, Cull, and Hollaway [3] derive an expression for the average proportion, $P_{1's}$, of bits that are 1, when the number n of bits is large. Table 2 shows this. In the Zeckendorf numeration system using Fibonacci numbers ($m = 2$), the average proportion of 1's is near 25%. However, as m increases from 2, this value approaches 50%.

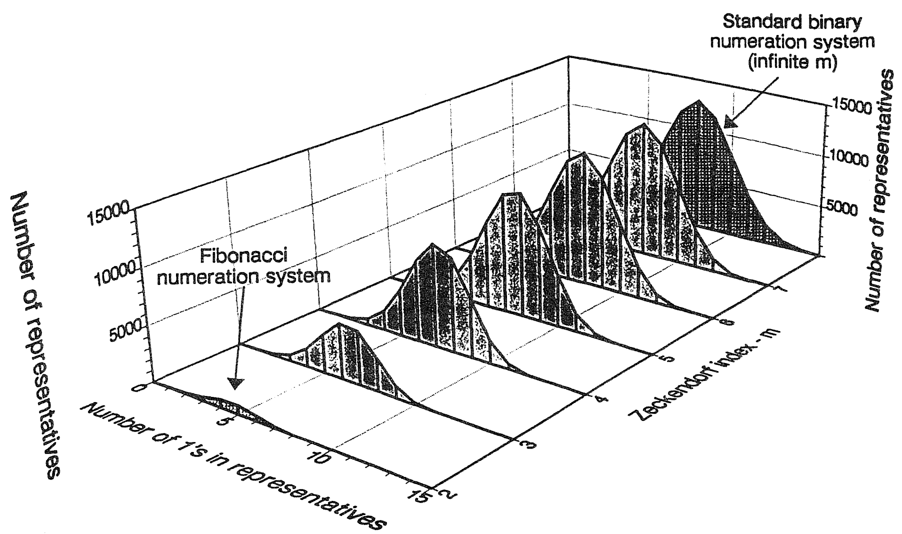

FIGURE 1. Distributions of 1's in 16-Tuple Zeckendorf Numeration Systems

TABLE 2 ([3], [11]). Average Proportion of 1's in Numeration Systems with Basis Elements $F_i = F_{i-1} + F_{i-2} + \cdots + F_{i-m}$ When the Number of Bits Is Large

m	2	3	4	5	6	7	8	∞
$P_{1's}$	0.2764	0.3816	0.4337	0.4621	0.4782	0.4875	0.4929	0.5000

Klein [11] considers numeration systems based on the recurrence $F_i = F_{i-1} + F_{i-m}$ for $i > m+1$, and $F_i = i-1$ for $1 < i \le m+1$, where $m \ge 2$. Consider a representative n-tuple $\boldsymbol{T}$ = $(a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$, where $a_i \in \{0, 1\}$. From Theorem 1 in [6] it follows that, if every pair of 1's is separated by at least $m-1$ 0's, then T is a unique representation of $N=\sum_{i=0}^{n-1} a_i F_{i+2}$. For $m=2$, this is the Fibonacci numeration system in which no two 1's are adjacent. A regular expression for the allowed representatives is

$$\mathbf{R}=\mathbf{0}^*+(\mathbf{0}+\mathbf{10}^{m-1})^*\,\mathbf{10}^*. \tag{4}$$

The $\mathbf{1}$ in $\mathbf{10}^*$ represents the rightmost 1 in a string containing at least one 1. In this case, any number of 0's, as described by $\mathbf{0}^*$ occurs to its right. $(\mathbf{0}+\mathbf{10}^{m-1})^*$ represents a string consisting of a sequence of substrings of the form 0 and 10^{m-1}. It follows from this construction that each pair of 1's is separated by at least $m-1$ 0's.

Consider a generating function $N(x,y,z)$ to count the representative n-tuples and the 0's and 1's in these representatives. From (4), we can write

$$\begin{aligned} N(x,y,z) &= (1+xy+x^2y^2+\cdots) \\ &\quad +[(1+(xy+x^my^{m-1}z)+(xy+x^my^{m-1}z)^2+\cdots)\,xz\,(1+xy+x^2y^2+\cdots)]. \end{aligned} \tag{5}$$

Here, $(1+xy+x^2y^2+\cdots)$ counts the ways to choose no 0's, one 0, two 0's, and so forth, while $(xy+x^my^{m-1}z)$ counts the ways to choose either a single 0 or 10^{m-1}, and xz counts the choice of a single 1. Equivalent to (5) is the following:

$$N(x,y,z)=\frac{1}{1-xy}\left[1+\frac{xz}{1-xy-x^my^{m-1}z}\right]. \tag{6}$$

From this, we obtain the distribution of 16-tuples according to the number of 1's, as shown in Figure 2. It is interesting that, even for small m, the number of representative n-tuples is small compared to the standard binary numeration system, shown here truncated to 1000 in order to display detail.

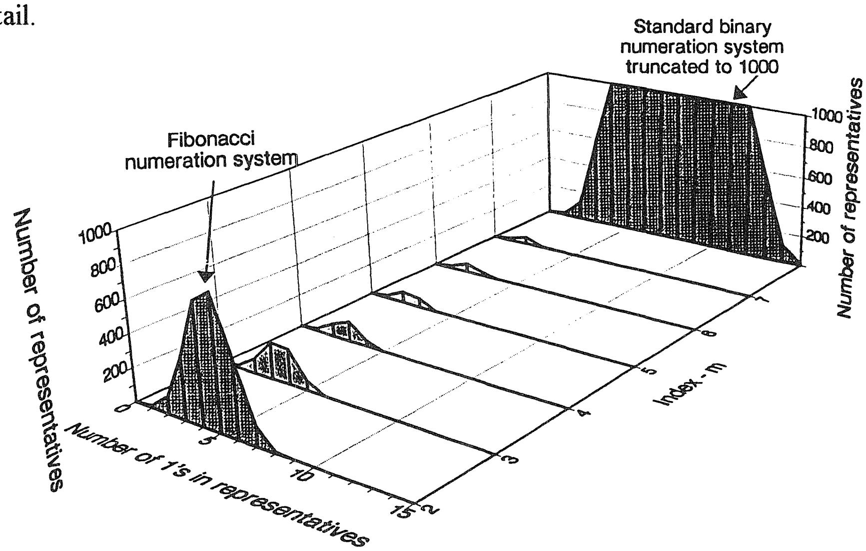

FIGURE 2. Distributions of 1's in 16-Tuple Numeration Systems Whose Basis Elements Are Generated by the Recurrence $F_i=F_{i-1}+F_{i-m}$

If we substitute 1 for y and z in (6), we achieve a generating function for the number of representative n-tuples, as follows:

$$N(x,1,1)=\frac{1}{1-x}\left[1+\frac{x}{1-x-x^m}\right]=\frac{1-x^m}{(1-x)(1-x-x^m)}. \tag{7}$$

Specifically, $\xi_{n\square\square}$, the coefficient of x^n in the power series representation of (7), is the number of representative n-tuples. We can write (7) as

$$N(x,1,1)=\frac{g_m}{1-\frac{x}{\alpha_m}}+\cdots, \tag{8}$$

where ... represents terms whose contribution to $\xi_{n\square\square}$ is negligible, for large n, compared to the term shown, and

$$g_m=\frac{1}{(1-\alpha_m)(a+m\alpha_m^{m-1})}.$$

Here, α_m is the dominant root, i.e., the singularity on the circle of convergence of $N(x,1,1)$. We are interested in the value of $\xi_{n\square\square}$ when n is large; thus, we write

$$\xi_{n\square\square}\sim\frac{1}{(1-\alpha_m)(a+m\alpha_m^{m-1})}\left(\frac{1}{\alpha_m}\right)^n, \tag{9}$$

where $f_n\sim g_n$ means $\lim_{n\to\infty}\frac{f_n}{g_n}=1$.

Consider now the proportion of bits that are 0 and 1 in the representatives counted by $\xi_{n\square\square}$. Substituting 1 for z in (6) yields $N(x,y,1)$, a generating function in which a typical term is $(\xi_{n0\square}+\xi_{n1\square}y^1+\xi_{n2\square}y^2+\cdots+\xi_{nn\square}y^n)x^n$, where $\xi_{ni\square}$ is the number of representative n-tuples with i 0's. Differentiating $N(x,y,1)$ with respect to y and setting $y=1$ yields a generating function in x in which a typical term is $(\xi_{n1\square}+2\xi_{n2\square}+n\xi_{nn\square})x^n=\Xi_n x^n$. Dividing Ξ_n by $\xi_{n\square\square}$ yields the average number of 0's in representative n-tuples. Dividing this by n gives the average proportion $P_{0\text{'s}}$ of bits that are 0. That is,

$$N_{0\text{'s}}(x)=\sum_{n\geq 0}\Xi_n x^n=\frac{d}{dy}N(x,y,1)\bigg|_{y=1}=\frac{(1-x^m)(x+(m-1)x^m)}{(1-x)(1-x-x^m)^2}+\cdots, \tag{10}$$

where $\cdots$ represents terms whose contribution to Ξ_n is negligible for large n, compared to the term shown. But $N_{0\text{'s}}(x)$ can be expressed as

$$N_{0\text{'s}}(x)=\frac{g_m^2(1-\alpha_m)(1+(m-1)\alpha_m^{m-1})}{\left(1-\frac{x}{\alpha_m}\right)^2}+\cdots, \tag{11}$$

where ... represents negligible terms. Therefore, from (11), we have

$$\Xi_n\sim g_m^2(1-\alpha_m(1+(m-1)\alpha_m^{m-1})\left(\frac{1}{\alpha_m}\right)^n n.$$

Thus, the proportion of digits that are 0 when the number n of digits is large is

$$P_{0\text{'s}}=\frac{1+(m-1)\alpha_m^{m-1}}{1+m\alpha_m^{m-1}}.$$

Table 3 summarizes these results. It includes $P_{1's}$, the proportion of bits that are 1, which can be obtained from $P_{1's} = 1 - P_{0's}$. It also shows values for the proportion of 0's and 1's for large n. As m grows, the proportion of bits that are 1 approaches 0, as shown in the last row of the table. All entries in this row are approximations that apply when m is large. For example, the dominant root α_m of $1-x-x^m$ can be calculated as follows. Let $\alpha_m = e^{-\Delta}$. For small Δ, this can be approximated be the truncated series $1-\Delta$. Thus, $1-\alpha_m-\alpha_m^m = 0 \approx \Delta - e^{-m\Delta}$. For $t = m\Delta$, we find that $t \cong me^{-t}$, from which we obtain $\ln m \cong \ln t + t \cong t$. Thus, $1-\alpha_m-\alpha_m^m \approx 1-\alpha_m-\Delta \approx 1-\alpha_m - t/m$ $\approx 1-\alpha_m - \ln m/m$ or $\alpha_m \cong 1-(\ln m)/m$. By a similar calculation, the approximation for the number of representative n-tuples shown in the last row, second column of Table 3 can be derived.

TABLE 3. Asymptotic Approximations to the Number of Representative n-Tuples and the Proportion of 0's and 1's in Numeration Systems with Basis Elements $F_i = F_{i-1} + F_{i-m}$

m	Number of representative n-tuples	Proportion of bits that are 0	Proportion of bits that are 1	α_m
General m	$\frac{1}{(1-\alpha_m)(1+m\alpha_m^{m-1})}\left(\frac{1}{\alpha_m}\right)^n$	$\frac{(1+(m-1)\alpha_m^{m-1})}{(1+m\alpha_m^{m-1})}$	$\frac{(1-\alpha_m)}{\alpha_m(1+m\alpha_m^{m-1})}$	Dominant root of $1-x-x^m$
2	1.1708×1.6180^n	0.7236	0.2764	0.6180
3	1.3134×1.4656^n	0.8057	0.1943	0.6823
4	1.4397×1.3803^n	0.8492	0.1508	0.7245
5	1.5550×1.3247^n	0.8762	0.1238	0.7549
6	1.6621×1.2852^n	0.8948	0.1052	0.7781
7	1.7630×1.2554^n	0.9084	0.0916	0.7965
8	1.8587×1.2320^n	0.9188	0.0812	0.8117
$\to\infty$	$m/\ln^2 m$	$1 - 1/m$	$1/m$	$1-(\ln m)/m$

3. MULTIPLE-VALUED NUMERATION SYSTEMS

There has been less work on numeration systems with nonbinary digits. Klein [11] considers numeration systems based on the recurrence $F_i = mF_{i-1} - F_{i-2}$ for $i > 3$, $F_3 = m$, and $F_2 = 1$, where $m \geq 3$. Consider a representative n-tuple $T = (a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$, where $a_i \in \{0, 1, \ldots, m-1\}$. From [11], if every pair of $m-1$'s is separated by at least one i, such that $i \in \{0, 1, \ldots, m-3\}$, then T is a unique representative of $N = \sum_{i=0}^{n-1} a_i F_{i+2}$. For this numeration system, we seek the proportion of digits that are $0, 1, \ldots, m-2$ and $m-1$. We use a generating function $N(x, y, z, w)$ in which x tracks the number of digits, y tracks the number of $m-1$'s, z tracks the number of $m-2$'s, and w tracks the number of 0's. By symmetry, the proportion of digits that are i, where i is restricted by $1 \leq i \leq m-3$, is the same as the proportion of 0's. Indeed, w can be viewed as tracking any i in the range $0 \leq i \leq m-3$.

We enumerate a representative according to whether it has 1) no $m-1$'s or 2) at least one $m-1$. For 1), there is no restriction on the digits, and the representatives are described by the regular expression, $\mathbf{P} = (\mathbf{0} + \mathbf{1} + \mathbf{2} + \cdots + \boldsymbol{m} - \mathbf{2})^*$. The power series expression for the number of representatives, in this case, is

$$1 + (wx + (m-3)x + zx) + (wx + (m-3)x + zx)^2 + (wx + (m-3)x + zx)^3 + \cdots. \tag{12}$$

That is, the term wx represents a choice of a 0 that contributes 1 to the count of 0's, as tracked by w, and 1 to the count of digits, as tracked by x. Similarly, the term zx tracks the number of

$m-2$'s. The term $(m-3)x$ tracks the number of digits in $\{1, 2, \ldots, m-3\}$. Expression (12) can be written as

$$\frac{1}{1-wx-(m-3)x-zx}. \tag{13}$$

For 2), the regular expression that describes the allowed representatives is

$$[\mathbf{P}+(\boldsymbol{m}-\mathbf{1})(\boldsymbol{m}-\mathbf{2})^*(\mathbf{0}+\mathbf{1}+\mathbf{2}+\cdots+(\boldsymbol{m}-\mathbf{3}))]^*(\boldsymbol{m}-\mathbf{1})\mathbf{P}.$$

Here, the rightmost $\boldsymbol{m}-\mathbf{1}$ is the rightmost $m-1$ in the string. To its right is any substring consisting of the digits 0, 1, ..., and $m-2$, as described by $\mathbf{P}$ and enumerated by (13). The digits to the left of the rightmost $m-1$ can be chosen from 0, 1, 2, ..., $m-2$ (i.e., from $\mathbf{P}$) and from strings beginning in $m-1$, ending in a digit whose value is $m-3$ or less with no, one, two, etc. $m-2$'s in between. The choices for the digits to the left of the rightmost $m-1$ are enumerated by

$$1+\left[wx+(m-3)x+zx+\frac{yx[wx+(m-3)x]}{1-zx}\right]+\left[wx+(m-3)x+zx+\frac{yx[wx+(m-3)x]}{1-zx}\right]^2+\cdots.$$

Here, the choices of a substring beginning in $m-1$ are enumerated by $yx[wz+(m-3)x]/(1-zx)$, where yz represents the choice of the first digit $m-1$, $[wz+(m-3)x]$ represents choice of the last digit, 0, 1, ..., $m-2$, and $1/(1-zx)$ represents the choice of the $m-2$'s in between. Thus, the generating function for the choices of representatives is

$$N(x,y,z,w)=\frac{1}{1-wx-(m-3)x-zx}\left[1+\frac{yx}{1-wx-(m-3)x-zx-\dfrac{yx[wx+(m-3)x]}{1-zx}}\right]. \tag{14}$$

Substituting 1 for y, z, and w into (14) yields $N(x,1,1,1)$, where

$$N(x,1,1,1)=\frac{1}{1-mx+x^2} \tag{15}$$

is the generating function for the number of representative n-tuples in this numeration system. Specifically, $\xi_{n\square\square\square}$, the coefficient of x_n in the power series representation of (15), is the number of representative n-tuples. We prefer to write (15) as

$$N(x)=\frac{g_m}{1-\frac{x}{\alpha_m}}+\frac{h_m}{1-\frac{x}{\beta_m}}, \tag{16}$$

where

$$\alpha_m=\frac{m-\sqrt{m^2-4}}{2},\ \beta_m=\frac{m+\sqrt{m^2-4}}{2}\left(=\frac{1}{\alpha_m}\right),\ g_m=\frac{1}{1-\alpha_m^2},\ \text{and}\ h_m=\frac{1}{1-\beta_m^2}.$$

That is, from (16), we can write $\xi_{n\square\square\square}=g_m(1/\alpha_m)^n+h_m(1/\beta_m)^n$. We are interested in the value of $\xi_{n\square\square\square}$ when n is large, so we use only the left term of the right side of (16). Thus,

$$\xi_{n\square\square\square}\sim\frac{1}{1-\alpha_m^2}\left(\frac{1}{\alpha_m}\right)^n. \tag{17}$$

Table 4 shows the values of g_m and $1/\alpha_m$ for various m.

TABLE 4. Asymptotic Approximations to the Number of Representative n-Tuples and Proportion of Digits in Numeration Systems with Basis Elements $F_i = mF_{i-1} - F_{i-2}$

m	Number of representative n-tuples	Proportion of digits that are i for $0 \le i \le m-3$	Proportion of digits that are $m-2$	Proportion of digits that are $m-1$	α_m
General m	$\frac{1}{1-\alpha_m^2}\left(\frac{1}{\alpha_m}\right)^n$	$\frac{\alpha_m}{1-\alpha_m^2}$	α_m	$\frac{\alpha_m(1-\alpha_m)}{(1+\alpha_m)}$	$\frac{m-\sqrt{m^2-4}}{2}$
3	1.1708×2.6180^n	0.4472	0.3820	0.1708	0.3820
4	1.0774×3.7321^n	0.2887	0.2679	0.1547	0.2679
5	1.0455×4.7913^n	0.2182	0.2087	0.1366	0.2087
6	1.0303×5.8284^n	0.1768	0.1716	0.1213	0.1716
7	1.0217×6.8541^n	0.1491	0.1459	0.1087	0.1459
8	1.0164×7.8730^n	0.1291	0.1270	0.0984	0.1270
$\to \infty$	$1.0000 \times m^n$	$1/m$	$1/m$	$1/m$	$1/m$

Substituting 1 for y and z in (14) yields $N(x,1,1,w)$. A typical term in the power series representation of this generating function is $(\xi_{n\square\square 0} + \xi_{n\square\square 1}w^1 + \xi_{n\square\square 2}w^2 + \cdots + \xi_{n\square\square n}w^n)x^n$, where $\xi_{n\square\square k}$ is the number of representative n-tuples with k 0's. Differentiating this with respect to w and setting $w = 1$ yields a generating function in x in which a typical term is $(\xi_{n\square\square 1} + 2\xi_{n\square\square 2} + \cdots + n\xi_{n\square\square n})x^n = \Xi_n x^n$. Dividing Ξ_n by $\xi_{n\square\square\square}$ yields the average number of 0's in representative n-tuples. Dividing this by n gives the average proportion of digits that are 0.

$$N_{0's}(x) = \sum_{n\ge 0} \Xi_n x^n = \frac{d}{dw} N(x,1,1,w)\bigg|_{w=1} = \frac{x}{(1-mx+x^2)^2}. \tag{18}$$

But $N_{0's}(x)$ can be expressed as

$$N_{0's}(x) = \frac{\dfrac{\alpha_m}{(1-\alpha_m^2)^2}}{\left(1-\dfrac{x}{\alpha_m}\right)^2} + \cdots, \tag{19}$$

where $\cdots$ represents terms whose contribution to Ξ_n is negligible, for large n, compared to the contributions from the term shown. Therefore, from (19), we have

$$\Xi_n \sim \frac{\alpha_m}{(1-\alpha_m^2)^2}\left(\frac{1}{\alpha_m}\right)^n n.$$

Thus, the proportion of digits that are 0 when the number of digits is large is $P_{0's} = \alpha_m / (1-\alpha_m^2)$. By an earlier observation, we can write $P_{m-3's} = \cdots = P_{1's} = P_{0's}$. Similarly, for the $m-2$'s, we have

$$N_{m-2's}(x) = \sum_{n\ge 0} \Xi_n x^n = \frac{d}{dz} N(x,1,z,1)\bigg|_{z=1} = \frac{(1-x^2)x}{(x^2-mx+1)^2} = \frac{\dfrac{\alpha_m}{1-\alpha_m^2}}{\left(1-\dfrac{x}{\alpha_m}\right)^2} + \cdots, \tag{20}$$

where $\cdots$ represents terms that can be neglected, when n is large. Therefore, from (20), we have

$$\Xi_n \sim \frac{\alpha_m}{1-\alpha_m^2}\left(\frac{1}{\alpha_m}\right)^n n \quad \text{and} \quad P_{m-2\text{'s}} = \alpha_m.$$

Table 4 above shows the various proportions. It includes an expression for $P_{m-1\text{'s}}$, which is obtained from $P_{m-1\text{'s}} = 1-(m-2)P_{0\text{'s}} - P_{m-2\text{'s}}$. Note that, as m grows, the proportion of digits that are i for $0 \le i \le m-1$ becomes nearly equal.

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