ON WEIGHTED r-GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

The standard Fibonacci numbers have several well-known and familiar properties, among which are the fact that the ratio of successive terms approaches a fixed limit ϕ, and that the n^{th} Fibonacci number is asymptotic to ϕ^n. In this paper we extend these properties to a generalized class of Fibonacci sequences, giving necessary and sufficient conditions for such a sequence to be asymptotic to one of the form $n^{\nu-1}\lambda^n$. In that case, we show how to compute the limiting ratio between the solution and $n^{\nu-1}\lambda^n$, as well as proving that the ratio of successive terms of a solution must have λ as a limit.

The necessary and sufficient conditions mentioned above are stated in terms of the roots of a polynomial. Indeed, this polynomial is the characteristic polynomial associated with the difference equation defining a generalized Fibonacci sequence. We also discuss conditions that depend directly on the coefficients of the characteristic polynomial. As a special case, we derive results when the polynomial has negative real coefficients (except for the leading coefficient 1). More generally, we give a sufficient condition on the coefficients for the roots to satisfy the necessary and sufficient conditions discussed above.

2. PRELIMINARIES

Let $a_1, a_2, \dots, a_r$ be arbitrary $r \geq 2$ complex numbers with $a_r \neq 0$, and let $A = (\alpha_{-r+1}, \alpha_{-r+2}, \dots, \alpha_{-1}, \alpha_0)$ be any given sequence of complex numbers. The *weighted r-generalized Fibonacci sequence* $\{y_A(n)\}_{n=-r+1}^{+\infty}$ is the sequence generated by the difference equation with initial values:

$$\begin{cases} y_A(n) = \alpha_n, & n = -r+1, -r+2, \ldots, -1, 0; \\ y_A(n) = \sum_{i=1}^{r} a_i y_A(n-i), & n = 1, 2, 3, \ldots. \end{cases} \tag{1}$$

As a special case, when $a_i = 1$ for all i (the unweighted case), $\alpha_0 = 1$, and $\alpha_i = 0$ for $i = -r+1, \ldots, -1$, (1) generates the *r-generalized Fibonacci numbers* introduced by Miles [8]. Explicit representations for these numbers can be found in [3], [5], and [6].

The polynomial $p(x) = x^r - a_1 x^{r-1} - \cdots - a_{r-1}x - a_r$ is called the *characteristic polynomial* associated to (1), and any solution λ of the *characteristic equation* $p(x) = 0$ is called a *characteristic root* for (1).

The first result, whose proof can be found in Kelley and Peterson [7], for example (or also in Jeske [4] or Ostrowski [9, §12]), relates the general solution of (1) to its characteristic roots.

Theorem 1: Suppose (1) has characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$, respectively $(m_1 + m_2 + \cdots + m_k = r)$. Then (1) has r independent solutions $n^j \lambda_\ell^n$ ($j = 0, \ldots, m_{\ell-1}$; $\ell = 1, \ldots, k$). Moreover, any solution of (1) is of the form

$$y_A(n) = \sum_{\ell=1}^{k} \sum_{j=0}^{m_\ell - 1} \beta_{\ell,j} n^j \lambda_\ell^n, \tag{2}$$

where the $\beta_{\ell,j}$ are determined by the initial condition $A = (\alpha_{-r+1}, \alpha_{-r+2}, \ldots, \alpha_{-1}, \alpha_0)$. □

Remark 2: Any independent solution $n^j \lambda_\ell^n$ ($j = 0, \ldots, m_{\ell-1}$; $\ell = 1, \ldots, k$) can be generated from the initial conditions $\alpha_i = i^j \lambda_\ell^i$ for $i = -r+1, \ldots, -1$ and $\alpha_0 = 1$. □

3. THE MAIN RESULTS

The necessary and sufficient conditions we consider are given in terms of the roots of the characteristic polynomial associated to (1). To simplify, we introduce the following terminology.

The polynomial $p(x)$ is called *asymptotically simple* if, among its roots of maximal modulus, there is a unique root λ of maximal multiplicity ν. Then λ is called the *dominant root* of $p(x)$ and ν is called the *dominant multiplicity*. Also, the system (1) is *asymptotically simple with dominant root λ and dominant multiplicity ν* if its characteristic polynomial is.

Theorem 3: System (1) is asymptotically simple with dominant root λ and dominant multiplicity ν if and only if, for any initial condition A, the sequence

$$\left\{ \frac{y_A(n)}{n^{\nu-1}\lambda^n} \right\}_{n=1}^{+\infty}$$

converges to a limit L_A, with L_A not equal to 0 for at least one A.

Proof: To prove the *if* part, we observe from (2) and Remark 2 that the convergence for all A implies the convergence of the sequence

$$\left\{ \frac{n^j}{n^{\nu-1}} \left(\frac{\lambda_\ell}{\lambda} \right)^n \right\}_{n=1}^{+\infty} \tag{3}$$

for any $\ell = 1, \ldots, k$ and $j = 0, \ldots, m_\ell - 1$. But, for each ℓ, the convergence of the sequence (3) for $j = 0, \ldots, m_\ell - 1$ implies that $|\lambda_\ell| < |\lambda|$ or $|\lambda_\ell| = |\lambda|$ and $\nu > m_\ell$ for $\lambda_\ell \neq \lambda$ or $\nu \geq m_\ell$ for $\lambda_\ell = \lambda$. Moreover, all the limits are zero except for $j = m_\ell - 1$ when $\lambda_\ell = \lambda$ and $\nu = m_\ell$. Also, the convergence to a nonzero limit L_A for some A implies that at least one sequence (3) has a nonzero limit. Hence, λ must be the dominant root and ν the dominant multiplicity. The *only if* part follows directly from Theorem 1. □

The next step is to relate $y_A(n)$ for arbitrary A to $y_0(n)$, the solution of (1) obtained for the initial conditions $\alpha_i = 0$, for $i = -r+1, \ldots, -1$, and $\alpha_0 = 1$. The matrix approach allows us to obtain the desired relation.

Let T be the (r, r)-matrix defined by

$$T = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_r \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

and $Y_A(n)$ be the $(r, 1)$-matrix defined by

$$Y_A(n) = \begin{bmatrix} y_A(n) \\ y_A(n-1) \\ \vdots \\ y_A(n-r+1) \end{bmatrix}, \quad n = 0, 1, 2, \ldots.$$

Hence, $Y_A(0) = A$ [if we consider A as an $(r, 1)$-matrix]. Therefore, we have $Y_A(n+1) = TY_A(n)$ and $Y_A(n) = T^n A$.

Let us also define the $(r, 1)$-matrices Y_i $(i = 0, 1, \ldots)$ by

$$Y_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow (i+1)^{\text{th}} \text{ entry}, \quad i = 0, 1, \ldots, r-1, \\ \\ \\ \\ \end{matrix}$$

and $Y_i = 0$ for $i = r, r+1, \ldots$. Let $Y_i(n) = T^n Y_i$ for $n = 0, 1, 2, \ldots$. Hence $TY_i(n) = Y_i(n+1)$, and also

$$Y_i(n) = \begin{bmatrix} y_i(n) \\ y_i(n-1) \\ \vdots \\ y_i(n-r+1) \end{bmatrix},$$

where $\{y_i(n)\}_{n=-r+1}^{+\infty}$ is the solution of (1) with the initial condition $A = Y_i$ ($\alpha_{-j} = 1$ if $j = i$ and $\alpha_{-j} = 0$ if $j \neq i$ for $j = 0, 1, \ldots, r-1$).

Since $A = \sum_{i=0}^{r-1} \alpha_{-i} Y_i$, it follows that

$$Y_A(n)=\sum_{i=0}^{r-1}\alpha_{-i}Y_i(n). \tag{4}$$

From these definitions and notation, we have the following direct result.

Lemma 4: Let $a_i=0$ for $i=r+1,r+2,\ldots$, then we have:

(a) $TY_i=a_{i+1}Y_0+Y_{i+1}$;

(b) for $i\geq 0$,

$$Y_i(n)=\sum_{j=1}^{n}a_{i+j}Y_0(n-j)+Y_{i+n},\quad n=0,1,2,\ldots;$$

(c) for any A,

$$Y_A(n)=\alpha_0Y_0(n)+\begin{cases}\displaystyle\sum_{i=1}^{r-1}\alpha_{-i}\sum_{j=1}^{n}a_{i+j}Y_0(n-j)+\sum_{i=1}^{r-1-n}\alpha_{-i}Y_{i+n}, & 0\leq n<r-1,\\ \displaystyle\sum_{i=1}^{r-1}\alpha_{-i}\sum_{j=1}^{r-i}a_{i+j}Y_0(n-j), & n\geq r-1;\end{cases} \tag{5}$$

(d) for $n\geq 0$, we have

$$y_A(n)=\alpha_0y_0(n)+\sum_{i=1}^{r-1}\alpha_{-i}\sum_{j=1}^{r-i}a_{i+j}y_0(n-j). \tag{6}$$

Proof: ***(a)*** is a direct consequence of the definitions and notation. ***(b)*** is easily obtained by induction. ***(c)*** follows by substitution of ***(b)*** into (4). We obtain ***(d)*** by considering the first entry in (5). □

The identity (6) leads to the next result.

Theorem 5: Let λ be any nonzero complex number and ν be a positive integer.

(a) The sequence

$$\left\{\frac{y_A(n)}{n^{\nu-1}\lambda^n}\right\}_{n=1}^{+\infty}$$

converges for any A (with limit L_A) if and only if the sequence

$$\left\{\frac{y_0(n)}{n^{\nu-1}\lambda^n}\right\}_{n=1}^{+\infty}$$

converges (with limit L_0). Moreover, the limits are related by the formula

$$L_A=\left[\alpha_0+\sum_{i=1}^{r-1}\alpha_{-i}\lambda^i\sum_{j=i}^{r-1}\frac{a_{1+j}}{\lambda^{1+j}}\right]L_0. \tag{7}$$

(b) Moreover, in (a), $L_A\neq 0$ for some A if and only if $L_0\neq 0$. In that case,

$$L_0=\left[\delta_{\nu 1}+(-1)^{\nu-1}\sum_{j=1}^{r-1}\frac{a_{1+j}}{\lambda^{1+j}}\sum_{i=1}^{j}i^{\nu-1}\right]^{-1}. \tag{8}$$

Proof: ***(a)*** From (6), we have

$$\frac{y_A(n)}{n^{\nu-1}\lambda^n} = \alpha_0 \frac{y_0(n)}{n^{\nu-1}\lambda^n} + \sum_{i=1}^{r-1} \alpha_{-i} \sum_{j=1}^{r-i} \frac{a_{i+j}}{\lambda^j} \left(\frac{n-j}{n}\right)^{\nu-1} \frac{y_0(n-j)}{(n-j)^{\nu-1}\lambda^{n-j}},$$

and (7) follows. For *(b)*, (1) must be asymptotically simple with dominant root λ and dominant multiplicity ν. Then, for $\alpha_0 = \delta_{\nu 1}, \alpha_{-i} = (-i)^{\nu-1}\lambda^{-i}$ $(i = 1, \ldots, r-1)$, we have $y_A(n) = n^{\nu-1}\lambda^n$ and $L_A = 1$ in (a). it follows that

$$L_0 = \left[\delta_{\nu 1} + (-1)^{\nu-1} \sum_{i=1}^{r-1} i^{\nu-1} \sum_{j=i}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}\right]^{-1},$$

and we get (8). □

Immediate consequences of these results are the next two theorems.

Theorem 6 (Ratio of weighted r-generalized Fibonacci sequences): Assume (1) asymptotically simple with dominant root λ and dominant multiplicity ν. If $A = (\alpha_{-r+1}, \alpha_{-r+2}, \ldots, \alpha_{-1}, \alpha_0)$ and $B = (\beta_{-r+1}, \beta_{-r+2}, \ldots, \beta_{-1}, \beta_0)$ are sequences of r complex numbers such that $L_B \neq 0$, then

$$\lim_{n\to+\infty} \frac{y_A(n)}{y_B(n)} = \frac{L_A}{L_B} = \frac{\alpha_0 + \sum_{i=1}^{r-1} \alpha_{-i}\lambda^i \sum_{j=1}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}}{\beta_0 + \sum_{i=1}^{r-1} \beta_{-i}\lambda^i \sum_{j=1}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}}. \quad \square$$

Theorem 7 (Ratio of consecutive terms): Assume (1) asymptotically simple with dominant root λ and dominant multiplicity ν. If $A = (\alpha_{-r+1}, \alpha_{-r+2}, \ldots, \alpha_{-1}, \alpha_0)$ is such that $L_A \neq 0$, then

$$\lim_{n\to+\infty} \frac{y_A(n+1)}{y_A(n)} = \lambda. \quad \square$$

This last result has already been obtained for the unweighted r-generalized Fibonacci numbers (see, e.g., [2] and [3]).

4. THE CASE OF NONNEGATIVE a_i's

In this section we assume the a_i's are nonnegative real numbers and $a_r > 0$. The following lemma is well known for the unweighted case (see, e.g., [9, §12] or [2, Lemma 2]) and is given without proof.

Lemma 8: There exists a unique real strictly positive characteristic root λ for (1). Moreover, λ is a simple characteristic root and all other characteristic roots of (1) have moduli $\leq \lambda$. □

Theorem 9: Let $a_1, \ldots, a_r$ be nonnegative real numbers with $a_r > 0$, and let λ be the unique positive real number of Lemma 8. Then the following are equivalent:

(a) (1) is asymptotically simple with dominant root λ and dominant multiplicity 1;

(b) the greatest common divisor $\mathrm{GCD}\{i \mid a_i > 0\} = 1$.

Proof: ***(a)⇒(b)***. Suppose $\mathrm{GCD}\{i \mid a_i > 0\} = d > 1$, then $p(x)$ is a polynomial in $y = x^d$, which has a unique greatest root $\lambda_d > 0$ from Lemma 8. Hence, the d^{th} roots of λ_d are all roots of

$p(x)$ with the same modulus as $\lambda_d^{1/d}$, which contradicts (a). For ***(b)⇒(a)***, see Ostrowski [9, Th. 12.2]. □

Example 10: For the unweighted case ($a_i = 1$ for all i), Lemma 8 implies (1) is asymptotically simple with dominant root λ and multiplicity 1, and we get:

(a) $$L_0 = \left[1 + \sum_{j=1}^{r-1} \frac{j}{\lambda^{1+j}}\right]^{-1};$$

(b) $$\frac{L_A}{L_B} = \frac{\alpha_0 + \sum_{i=1}^{r-1} \alpha_{-i}\lambda^i \sum_{j=i}^{r-1} \frac{1}{\lambda^{1+j}}}{\beta_0 + \sum_{i=1}^{r-1} \beta_{-i}\lambda^i \sum_{j=i}^{r-1} \frac{1}{\lambda^{1+j}}}.$$

Dence [1] obtained similar results in terms of all roots of $p(x)$. Here we obtain the result in terms of only the largest root λ. □

5. THE GENERAL CASE

For arbitrary complex number a_i's, we do not have a result similar to Theorem 9. In Theorem 9, the nonnegativity of the a_i's is important. The next result and example illustrate this fact.

Theorem 11: Assume (1) asymptotically simple with dominant root λ and dominant multiplicity ν, then $\mathrm{GCD}\{i \mid a_i \neq 0\} = 1$.

Proof: Similar to ***(a)⇒(b)*** of Theorem 9. □

Unfortunately, the condition $\mathrm{GCD}\{i \mid a_i \neq 0\} = 1$ is not a sufficient condition for the general case.

Example 12: Let $p(x) = (x-1)(x-i) = x^2 - (1+i)x + i$, where $i = \sqrt{-1}$. Then $a_i = 1+i$, $a_2 = -i$, and $\mathrm{GCD}\{i \mid a_i \neq 0\} = 1$. But the characteristic roots 1 and i have the same modulus and multiplicity, hence (1) is not asymptotically simple. □

The general problem is to find criteria which are equivalent to or imply that (1) is asymptotically simple. In the sequel, we consider one such criterion which could certainly be weakened.

Let λ be *a priori* any characteristic root of $p(x)$, and set

$$b_i = \sum_{j=i}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}, \quad i = 0, 1, 2, \ldots.$$

Then $b_0 = 1$ and $b_i = 0$ for $i \geq r$. Consider $\alpha_{-i} = \lambda^{-i}$ for $i = 0, \ldots, r-1$. Then, from (6),

$$\lambda^n = y_0(n) + \sum_{i=1}^{r-1} \lambda^{-i} \sum_{j=1}^{r-i} a_{i+j} y_0(n-j), \quad n \geq 0,$$

and

$$1 = \frac{y_0(n)}{\lambda^n} + \sum_{j=1}^{r-1} b_j \frac{y_0(n-j)}{\lambda^{n-j}}. \tag{9}$$

Let $z(n) = \frac{y_0(n)}{\lambda^n} - \frac{y_0(n-1)}{\lambda^{n-1}}$. We have $z(0) = 1$, and from (9),

$$z(n) = -\sum_{j=1}^{r-1} b_j z(n-j), \quad n = 1, 2, 3, \dots. \tag{10}$$

It follows that

$$\sum_{j=0}^{n} b_j z(n-j) = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n = 1, 2, 3, \dots. \end{cases}$$

We can now prove the following convergence result for the sequence $\{z(n)\}_{n=0}^{+\infty}$.

Lemma 13: If $\sum_{j=1}^{r-1} |b_j| < 1$, then $\{z(n)\}_{n=0}^{+\infty}$ converges to 0.

Proof: From (10), it follows that

$$|z(n)| \le \left[\sum_{j=1}^{r-1} |b_j|\right] \max\{|z(n-j)| : j = 1, \dots, r-1\}, \text{ for } n \ge 1.$$

From this inequality, the sequence $\{M_n = \max\{|z(j)| : j \ge n\}\}_{n=0}^{+\infty}$ is a decreasing sequence, and $M_n \le [\sum_{j=1}^{r-1} |b_j|] M_{n-r+1}$ for any $n \ge r-1$. Hence $\{M_n\}_{n=0}^{+\infty}$ converges to 0 and the result follows since $|z(n)| \le M_n$. □

From the definition, we have

$$\frac{y_0(n)}{\lambda^n} = \sum_{j=0}^{n} z(j) \quad \text{and} \quad \sum_{j=0}^{n} b_j = \sum_{j=0}^{r-1} b_j, \text{ for } n \ge r-1.$$

Let us consider the following product for $n \ge r-1$:

$$\begin{aligned} \frac{y_0(n)}{\lambda^n} \sum_{j=0}^{r-1} b_j &= \left(\sum_{j=0}^{n} z(j)\right)\left(\sum_{j=0}^{n} b_j\right) \\ &= \sum_{j=0}^{n} \sum_{k=0}^{j} b_k z(j-k) + \sum_{j=1}^{r-1} b_k \sum_{k=n-j+1}^{n} z(k) = 1 + \sum_{j=1}^{r-1} b_j \sum_{k=0}^{j-1} z(n-k). \end{aligned} \tag{11}$$

Taking the limit in (11), we get

$$\lim_{n \to +\infty} \frac{y_0(n)}{\lambda^n} = \left[\sum_{j=0}^{r-1} b_j\right]^{-1},$$

and we have proved the following result.

Theorem 14: Let λ be a characteristic root of $p(x)$ and set

$$b_i = \sum_{j=i}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}, \quad \text{for } i = 0, 1, \dots, r-1.$$

If

$$\sum_{j=1}^{r-1} |b_j| < 1, \tag{12}$$

then (1) is asymptotically simple with dominant root λ and dominant multiplicity 1. □

The next result illustrates that the condition (12) is satisfied in many cases.

Theorem 15: For any fixed sequence $\{a_i\}_{i=2}^r$ $(r \geq 2, a_r \neq 0)$ of complex numbers, there exists a positive real number R such that, for any a_1 with $|a_1| > R$, the sequence of a_i's satisfies the condition of Theorem 14 for a root λ of the characteristic polynomial $p(x)$.

Proof: Using Lemma 8, let R_1 be the unique positive root of the equation

$$x^r = |a_2|x^{r-2} + 2|a_3|x^{r-3} + \cdots + (r-1)|a_r|.$$

Let us note that we have

$$|a_2| + 2|a_3|R_1^{-1} + 3|a_4|R_1^{-2} + \cdots + (r-1)|a_r|R_1^{-r+2} = R_1^2.$$

Set $R = rR_1$. Suppose $|a_1| > R$ and let λ be any root of maximum modulus of $p(x)$. Since the sum of all the roots of $p(x)$ (with multiplicity) is equal to a_1, we see that λ must have a modulus greater than or equal to $|a_1|/r$. Thus, we have $|\lambda| > R_1$. Then

$$\begin{aligned}\sum_{i=1}^{r-1}|b_i| &\leq \sum_{j=2}^{r}(j-1)|a_j||\lambda|^{-j}\\ &< |\lambda|^{-2}\sum_{j=2}^{r}(j-1)|a_j||\lambda|^{-j+2}\\ &= |\lambda|^{-2}R_1^2 < 1.\end{aligned}$$

Thus, the condition of Theorem 14 is satisfied. □

Remark 16: The last result is intuitively clear. It indicates that, if we increase $|a_1|$, eventually there will be only one root of maximum modulus. But increasing $|a_1|$ means that $p(x)$ behaves like $q(x) = x^r - a_1r^{r-1} = x^{r-1}(x - a_1)$. Moreover, increasing $|a_1|$ implies the modulus of the largest characteristic root increases also, and since the expression $\sum_{j=1}^{r-1}|b_j|$ does not contain a_1, it will eventually be less than 1 because each term contains negative powers of $|\lambda|$. □

Example 17: As an explicit example, consider the case $r = 3, a_1 = a, a_2 = i$, and $a_3 = -ai$, where $i = \sqrt{-1}$ and a is a complex number with $|a| > 1$. In this case, $|a| > 1$ is one of the characteristic roots of the polynomial $p(x) = (x-a)(x^2 - i)$, and we can check that $\sum_{j=1}^{r-1}|b_j| = 1/|a|^2$, which implies that the condition in Theorem 14 is satisfied. Hence, $\lim_{n\to+\infty}[y_0(n)]/a^n$ exists and is equal to $a^2(a^2+i)/(a^4+1)$. Note that when $|a| \leq 1$ the condition in Theorem 14 is not satisfied and the limit does not exist. □

Remark 18: For the nonnegative a_i's, condition (12) is useless because it implies $a_1 > 0$. Indeed, we have, for λ as given by Lemma 8,

$$\sum_{j=1}^{r-1}b_j = \sum_{i=1}^{r-1}\frac{ia_{i+1}}{\lambda^{i+1}} = \sum_{i=1}^{r-1}\frac{(i-1)a_{i+1}}{\lambda^{i+1}} + 1 - \frac{a_1}{\lambda}$$

and if $a_1 = 0$ then $\sum_{j=1}^{r-1}b_j \geq 1$. □

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