ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-826 ***Proposed by the editor***

Find a recurrence consisting of positive integers such that each positive integer n occurs exactly n times.

B-827 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland***

Find a solution to the recurrence $A_{n+3} = A_n - 2A_{n+2}$, $A_0 = 0$, $A_1 = 1$, $A_2 = -2$, in terms of F_n and L_n.

B-828 ***Proposed by Piero Filipponi, Rome, Italy***

For n a positive integer, prove that $\displaystyle\sum_{r=0}^{\lfloor\frac{n-1}{4}\rfloor}\binom{n-1-2r}{2r}$ is within 1 of $F_n/2$.

B-829 ***Proposed by Jack G. Segers, Liège, Belgium***

For n a positive integer, let $P_n = F_{n+1}F_n$, $A_n = P_{n+1} - P_n$, $B_n = A_n - A_{n-1}$, $C_n = B_{n+1} - B_n$, $D_n = C_n - C_{n-1}$, and $E_n = D_{n+1} - D_n$. Show that $|P_n - B_n|$, $|A_n - C_n|$, $|B_n - D_n|$, and $|C_n - E_n|$ are successive powers of 2.

B-830 ***Proposed by Al Dorp, Edgemere, NY***

(a) Prove that if $n = 84$ then $(n+3)|F_n$.
(b) Find a positive integer n such that $(n+19)|F_n$.
(c) Is there an integer a such that $n+a$ never divides F_n?

SOLUTIONS

Mr. Feta's Lost Theorem

B-808 ***Proposed by Paul S. Bruckman, Jalmiya, Kuwait***
(Vol. 34, no. 2, May 1996)

Years after Mr. Feta's demise at Bellevue Sanitarium, a chance inspection of his personal effects led to the discovery of the following note, scribbled in the margin of a well-worn copy of Professor E. P. Umbugio's "22/7 Calculated to One Million Decimal Places":

> To divide "n-choose-one" into two other non-trivial "choose one's", "n-choose-two", or in general, "n-choose-m" into two non-trivial "choose-m's", for any natural m is always possible, and I have assuredly found for this a truly wonderful proof, but the margin is too narrow to contain it.

Because of the importance of this result, it has come to be known as Mr. Feta's Lost Theorem. We may restate it in the following form:

Solve the Diophantine equation $x^{\underline{m}} + y^{\underline{m}} = z^{\underline{m}}$, for $m \le x \le y \le z, m = 1, 2, 3, \ldots$, where $X^{\underline{m}} = X(X-1)(X-2)\cdots(X-m+1)$. Was Mr. Feta crazy?

Solution by Gerald A. Heuer, Concordia College, Moorhead, MN & Karl W. Heuer, The Free Software Foundation, Cambridge, MA

Well, owning a copy of Umbugio's "22/7 Calculated . . ." perhaps casts a slight shadow over Mr. Feta, but his statement is correct. Presumably the book had extremely small margins, for the solution, $x = y = 2m-1$, $z = 2m$ for every natural number m, does not require a substantial margin.

For each of the cases $m = 1$ and $m = 2$ there are infinitely many solutions, and we may give the general solution. With $m = 1$, things are rather simple: $x \le y$ arbitrary, and $z = x + y$.

With $m = 2$, we have the following family of solutions: Choose $x \ge 3$ arbitrary, and choose integers s, t, u, v even, $uv > st$, and $su \le (uv - st + 1)/2$. Then one routinely verifies that

$$\left(su, \frac{uv - st + 1}{2}, \frac{uv + st + 1}{2}\right) \tag{1}$$

is a solution. Note that for every x the choice of $s = t = 1$ satisfies the remaining conditions, so at least one solution exists. Moreover, every solution is of this form, for if $x(x-1) + y(y-1) = z(z-1)$ and

$$z = y + r, \tag{2}$$

then one finds at once that

$$x(x-1) = r(r + 2y - 1), \tag{3}$$

so every prime factor of r divides either x or $x-1$. Thus, we may write $r = st$, where $s | x$ and $t | (x-1)$. Then u and v may be defined satisfying $x = su$ and $x - 1 = tv$, and solving (3) for y and using (2), we obtain the solution (1). That just one of s, t, u, v is even follows from the facts that x and $x-1$ have opposite parity and that y is an integer, so that $uv - st$ is odd. The inequalities assumed must hold in order that $2 \le x \le y$.

With $m = 3$, in addition to the above solution, we find $(10,16,17)$, $(22,56,57)$, $(36,120,121)$ and there seem to be no more with $z = y + 1$, but we do not have a proof. A computer search with $m = 3$ yields many solutions and suggests probably an infinite family exists. With $m = 4$ there is at least one more solution, $(132,190,200)$, and with $m = 6$ at least two more: $(14,15,16)$ and

$(19,19,21)$. Each of these two seems to begin an infinite family: Whenever m is a solution of the Fermat-Pell equation $8m^2+1=n^2$ (i.e., $m=6,35,204,\ldots$), we have a solution

$$\left(\frac{4m+n-3}{2}, \frac{4m+n-3}{2}, \frac{4m+n+1}{2}\right),$$

and whenever m is a solution of $5m^2-2m+1=n^2$ ($m=6,40,273,1870,\ldots$),

$$\left(\frac{3m+n-3}{2}, \frac{3m+n-1}{2}, \frac{3m+n+1}{2}\right)$$

is a solution.

Also solved by Leonard A. G. Dresel, Piero Filipponi, David E. Manes, H.-J. Seiffert, and the proposer.

It Keeps on Going

B-809 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland (Vol. 34, no. 2, May 1996)***

Let k be a fixed positive integer. Find a recurrence consisting of positive integers such that each positive integer occurs exactly k times.

Solution 1 by David E. Manes, SUNY College at Oneonta, NY; H.-J. Seiffert, Berlin, Germany; Lawrence Somer, The Catholic University of America, Washington, DC; and David Zeitlin, Minneapolis, MN (independently)

$$w_{n+2k}=2w_{n+k}-w_n, \quad n\ge 0,$$

with initial conditions $w_0=w_1=w_2=\cdots=w_{k-1}=1$; $w_k=w_{k+1}=w_{k+2}=\cdots=w_{2k-1}=2$.

Solution 2 by the proposer

$$w_{n+k+1}=w_{n+k}+w_{n+1}-w_n, \quad n\ge 0,$$

with initial conditions $w_0=w_1=w_2=\cdots=w_{k-1}=1$; $w_k=2$.

Solution 3 by Gerald A. Heuer, Concordia College, Moorhead, MN, and Russell Jay Hendel, Drexel University, Philadelphia, PA (independently)

$$w_{n+k}=w_n+1, \quad n\ge 0,$$

with initial conditions $w_0=w_1=w_2=\cdots=w_{k-1}=1$.

Solution 4 by Murray S. Klamkin, University of Alberta, Canada

$$w_{n+1}=1+\left\lfloor\frac{w_n+n}{k+1}\right\rfloor, \quad n>0,$$

with initial condition $w_1=1$.

Several solvers gave the reference G. Meyerson & A. J. van der Poorten, "Some Problems Concerning Recurrence Sequences," Amer. Math. Monthly 102 (1995):698-705, which contains related problems. See Problem B-826 in this issue for a related problem.

Also solved by Graham Lord.

Divisible Determinant

B-810 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 34, no. 2, May 1996)

Let $\langle H_n \rangle$ be a generalized Fibonacci sequence defined by $H_{n+2} = H_{n+1} + H_n$ for $n > 0$ with initial conditions $H_1 = a$ and $H_2 = b$, where a and b are integers. Let k be a positive integer. Show that

$$A_n = \begin{vmatrix} H_n & H_{n+1} \\ H_{n+k+1} & H_{n+k+2} \end{vmatrix}$$

is always divisible by a Fibonacci number.

Proof by Steve Edwards, Southern College of Technology, Marietta, GA

We use equation (8) in [1]: $H_{n+m} = F_{m-1}H_n + F_m H_{n+1}$. Then

$$\begin{aligned} A_n &= H_n H_{n+k+2} - H_{n+1}H_{n+k+1} \\ &= H_n[F_{k+1}H_n + F_{k+2}H_{n+1}] - H_{n+1}[F_k H_n + F_{k+1}H_{n+1}] \\ &= F_{k+1}[H_n^2 - H_{n+1}^2] + H_n H_{n+1}[F_{k+2} - F_k] \\ &= F_{k+1}[H_n^2 - H_{n+1}^2] + H_n H_{n+1} F_{k+1} = F_{k+1}[H_n^2 - H_{n+1}^2 + H_n H_{n+1}]. \end{aligned}$$

Thus, A_n is always divisible by F_{k+1}.

Disproof by Russell Jay Hendel, Drexel University, Philadelphia, PA

Since $F_1 = 1$, every integer is divisible by a Fibonacci number. Thus, the proposer probably intended to ask us to show that A_n is always divisible by a Fibonacci number larger than 1. But, in that case, the proposition is false. If $a = 2$, $b = 5$, and $k = 1$, then $A_1 = -11$, which is not divisible by any Fibonacci number larger than 1.

A more correct statement of the problem would have been: "Show that A_n is always divisible by F_{k+1}."

Generalization by Pentti Haukkanen, University of Tampere, Tampere, Finland

Let $\langle G_n \rangle$ and $\langle H_n \rangle$ be any two generalized Fibonacci sequences satisfying $G_{n+2} = G_{n+1} + G_n$ and $H_{n+2} = H_{n+1} + H_n$ for $n > 0$. Vajda [1, p. 27] proves that

$$G_{n+k+1}H_{n+h} - G_n H_{n+h+k+1} = F_{k+1}(G_{n+1}H_{n+h} - G_n H_{n+h+1}).$$

Thus,

$$\begin{vmatrix} G_n & H_{n+h} \\ G_{n+k+1} & H_{n+h+k+1} \end{vmatrix}$$

is always divisible by F_{k+1}.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Most solvers noted that $A_n = (-1)^{n-1}(a^2 - b^2 + ab)F_{k+1}$. Redmond and Somer showed (independently) that if $\langle W_n \rangle$ is a sequence of integers that satisfies the recurrence $W_{n+2} = PW_{n+1} + QW_n$ with initial conditions $W_1 = a$ and $W_2 = b$, then

$$J_k = \begin{vmatrix} W_n & W_{n+1} \\ W_{n+k+1} & W_{n+k+2} \end{vmatrix}$$

is equal to $U_{k+1}J_0$ *and, hence, is divisible by* U_{k+1}, *where* $\langle U_n\rangle$ *denotes the sequence satisfying the same recurrence as* $\langle W_n\rangle$ *with initial conditions* $U_0=0,\ U_1=1$.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Russell Jay Hendel, Murray S. Klamkin, Harris Kwong, Carl Libis, David Manes, Don Redmond, H.-J. Seiffert, Lawrence Somer, David Zeitlin, and the proposer.

Alternating Lucas

B-811 ***Proposed by Russell Euler, Maryville, MO***
(Vol. 34, no. 2, May 1996)

Let n be a positive integer. Show that:

(a) if $n\equiv 0 \pmod 4$, then $F_{n+1}=L_n-L_{n-2}+L_{n-4}-\cdots-L_2+1$;

(b) if $n\equiv 1 \pmod 4$, then $F_{n+1}=L_n-L_{n-2}+L_{n-4}-\cdots-L_3+1$;

(c) if $n\equiv 2 \pmod 4$, then $F_{n+1}=L_n-L_{n-2}+L_{n-4}-\cdots+L_2-1$;

(d) if $n\equiv 3 \pmod 4$, then $F_{n+1}=L_n-L_{n-2}+L_{n-4}-\cdots+L_3-1$.

Solution by L. A. G. Dresel, Reading, England

We use the well-known formula $F_{n-1}+F_{n+1}=L_n$, which is formula (6) in [1]. When n is even, consider the sum

$$\begin{aligned}S_n &= L_n-L_{n-2}+L_{n-4}-\cdots+(-1)^{(n-2)/2}L_2\\ &= (F_{n+1}+F_{n-1})-(F_{n-1}+F_{n-3})+\cdots+(-1)^{(n-2)/2}(F_3+F_1)\\ &= F_{n+1}+(-1)^{(n-2)/2}F_1.\end{aligned}$$

Since $F_1=1$, this proves (a) and (c).

When n is odd, $n\geq 3$, consider the sum

$$\begin{aligned}T_n &= L_n-L_{n-2}+L_{n-4}-\cdots+(-1)^{(n-3)/2}L_3\\ &= (F_{n+1}+F_{n-1})-(F_{n-1}+F_{n-3})+\cdots+(-1)^{(n-3)/2}(F_4+F_2)\\ &= F_{n+1}+(-1)^{(n-3)/2}F_2.\end{aligned}$$

Since $F_2=1$, this proves (b) and (d).

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Also solved by Paul S. Bruckman, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Harris Kwong, Carl Libis, David E. Manes, Bob Prielipp, Don Redmond, H.-J. Seiffert, Lawrence Somer, and the proposer.

A Triangle in Space

B-812 ***Proposed by John C. Turner, University of Waikato, Hamilton, New Zealand***
(Vol. 34, no. 2, May 1996)

Let $P,\ Q,\ R$ be three points in space with coordinates $(F_{n-1},0,0)$, $(0,F_n,0)$, $(0,0,F_{n+1})$, respectively. Prove that twice the area of ΔPQR is an integer.

Editorial composite of solutions received from Steve Edwards, Southern College of Technology, Marietta, GA, and Murray S. Klamkin, University of Alberta, Alberta, Canada

We will show that if $P=(x,0,0)$, $Q=(0,y,0)$, and $R=(0,0,z)$, where x, y, and z are positive integers such that $x+y=z$, then twice the area of ΔPQR is an integer.

Heron's Formula [1, p. 12] gives the area of a triangle with sides of lengths a, b, and c as $A=\frac{1}{2}\sqrt{s(s-a)(s-b)(s-c)}$, where $s=(a+b+c)/2$. Using the Pythagorean Theorem to get the sides of ΔPQR, and a straightforward algebraic reduction, gives

$$\begin{aligned}2A&=\sqrt{x^2y^2+y^2z^2+z^2x^2}=\sqrt{x^2y^2+y^2(x+y)^2+(x+y)^2x^2}\\&=\sqrt{x^4+2x^3y+3x^2y^2+2xy^3+y^4}=x^2+xy+y^2.\end{aligned}$$

Thus, $2A$ is an integer.

Reference

1. H. S. M. Coxeter. *Introduction to Geometry.* 2nd ed. New York: Wiley & Sons, 1989.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Harris Kwong, David E. Manes, John Oman & Bob Prielipp, H.-J. Seiffert, Lawrence Somer, David Zeitlin, and the proposer.

A Very General Determinant

B-813 ***Proposed by Peter Jeuck, Mahwah, NJ***
(Vol. 34, no. 2, May 1996)

Let $\langle X_n\rangle$, $\langle Y_n\rangle$, and $\langle Z_n\rangle$ be three sequences that each satisfy the recurrence $W_n=pW_{n-1}+qW_{n-2}$ for $n>1$, where p and q are fixed integers. (The initial conditions need not be the same for the three sequences.) Let a, b, and c be any three positive integers. Prove that

$$\begin{vmatrix}X_a & X_b & X_c\\ Y_a & Y_b & Y_c\\ Z_a & Z_b & Z_c\end{vmatrix}=0.$$

Solution by Paul S. Bruckman, Highwood, IL

Let $\langle U_n\rangle$ be the sequence that satisfies the same recurrence, but with initial values $U_0=0$ and $U_1=1$. The sequence $\langle qX_1U_{n-2}+X_2U_{n-1}\rangle$ also satisfies this same recurrence and has the same values as $\langle X_n\rangle$ when $n=1$ and $n=2$. Hence, these sequences are identical. In a similar manner, we see that $Y_n=qY_1U_{n-2}+Y_2U_{n-1}$ and $Z_n=qZ_1U_{n-2}+Z_2U_{n-1}$. Thus,

$$\begin{pmatrix}X_a & X_b & X_c\\ Y_a & Y_b & Y_c\\ Z_a & Z_b & Z_c\end{pmatrix}=\begin{pmatrix}qX_1 & X_2 & 0\\ qY_1 & Y_2 & 0\\ qZ_1 & Z_2 & 0\end{pmatrix}\begin{pmatrix}U_{a-2} & U_{b-2} & U_{c-2}\\ U_{a-1} & U_{b-1} & U_{c-1}\\ 0 & 0 & 0\end{pmatrix}.$$

Clearly, the determinant of each matrix on the right is 0. Hence, the determinant of the matrix on the left is 0.

Also solved by Leonard A. G. Dresel, Russell Jay Hendel, Murray S. Klamkin, Harris Kwong, David E. Manes, H.-J. Seiffert, Lawrence Somer, and the proposer.

Note: The Elementary Problems Column is in need of more *easy*, yet elegant and nonroutine problems.