ON SOME BASIC LINEAR PROPERTIES OF THE SECOND-ORDER INHOMOGENEOUS LINE-SEQUENCE

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(*Submitted April 1995*)

1. INTRODUCTION

A second-order line-sequence is inhomogeneous if its recurrence relation includes a nonzero constant k, such as the following:

$$u_n = u_{n-2} + u_{n-1} + 1, \tag{1.1}$$

where $k = 1$ is the inhomogeneous term.

A line-sequence generated by (1.1), according to the convention adopted in (2.1) of [3], is represented by

$$I_{u_0,u_1} : \ldots u_{-2}, u_{-1}, [u_0, u_1], u_2, u_3, \ldots, \tag{1.2}$$

where u_n is the n^{th} element counting from u_0 in both directions, and the pair u_0, u_1 is referred to as a generating pair. The algebraic properties of these sequences have been investigated by Bicknell and Bergum [1], and the general solution of an arbitrary order inhomogeneous sequence has been obtained by Liu [6]. In this article we investigate some basic linear properties of these line-sequences. We shall first treat the simple case of line-sequences generated by (1.1) in some detail. Later on we shall extend the treatment to more general cases.

Some samples of the inhomogeneous line-sequences given by (1.1) are:

$$I_{0,-1} : \ldots 4, -4, 1, -2, [0, -1], 0, 0, 1, 2, \ldots; \tag{1.3}$$

$$I_{-1,0} : \ldots -4, 1, -2, 0, [-1, 0], 0, 1, 2, 4, \ldots. \tag{1.4}$$

For reasons to be explained later, we say that these constitute an inhomogeneous Fibonacci pair. Also, for convenience, the terms of the line-sequences will be represented by

$$I_n : \ldots I_{-3}, I_{-2}, I_{-1}, I_0, I_1, I_2, I_3, \ldots, \tag{1.5}$$

where $I_0 = -1$ is the origin, $I_1 = 0$, and so on.

We shall call the change of a sequential relation from the homogeneous case to the corresponding inhomogeneous case an *inhomogeneous transformation*, and those relations that remain unchanged in form (*inhomogeneously*) *covariant*. As it turns out, many well-known sequential relations are found to be inhomogeneous covariant.

2. THE INHOMOGENEOUS HARMONIC CASE

We define the following inhomogeneous operations in relation to (1.1) and (1.2).

Definition 1: Addition is defined to be addition of corresponding numbers in the line-sequences, together with the inhomogeneous constant 1. Thus,

$$I_{i,j} = I_{i',j'} + I_{i'',j''}, \tag{2.1}$$

where $i = i' + i'' + 1$ and $j = j' + j'' + 1$. We refer to this operation as *inhomogeneous addition*.

Definition 2: Multiplication by a scalar h is defined in the sense of repeated addition. Thus,

$$I_{i,j} = hI_{i',j'}, \tag{2.2}$$

where h is a scalar, $i = hi' + h - 1$ and $j = hj' + h - 1$. We refer to this operation as *inhomogeneous multiplication*.

Definition 3: The inner product of two line-sequences is defined as follows:

$$(I_{i,j}, I_{i',j'}) = (i+1)(i'+1) + (j+1)(j'+1). \tag{2.3}$$

Two line-sequences are said to be orthogonal if and only if their inner product is zero, normal if and only if one's self inner product is one. The length of a line-sequence is defined as the (positive) square root of its inner product with itself.

Definition 4: Two line-sequences are said to be congruent if and only if they constitute the same set of numbers; equal if and only if they are congruent and have the same set of generating numbers. We refer to this as the *uniqueness of generating numbers*.

It is clear that the set I of line-sequences spans a vector space [2] referred to as an *inhomogeneous-harmonic (IH-)space*, where the first predicate signifies the type of operations and the second the recurrence relation. Furthermore, it can be verified easily that the line-sequences (1.3) and (1.4) form an orthonormal pair that serves as the basis set for this space. An arbitrary line-sequence in this space can then be resolved into its inhomogeneous basis components as follows:

$$I_{i,j} = (i+1)I_{0,-1} + (j+1)I_{-1,0}. \tag{2.4}$$

Applying (1.5), this equation can also be expressed in terms of I_n's:

$$I_{i,j} = (i+1)I_{I_{-1},I_0} + (j+1)I_{I_0,I_1}. \tag{2.5}$$

Following are some examples illustrating the inhomogeneous operations.

Example 1: Let $I_{a,b}$ be the identity element of addition. Then, for an arbitrary line-sequence $I_{i,j}$, we must have $I_{a,b} + I_{i,j} = I_{i,j}$. By (2.1), we have $I_{a,b} + I_{i,j} = I_{a+i+1,b+j+1}$. Comparing the right-hand sides of these equations, we obtain $a = b = -1$. Hence, the additive identity is a sequence of -1's:

$$I_{-1,-1}: \ldots -1, -1, [-1, -1], -1, -1, \ldots. \tag{2.6}$$

Example 2: By (2.1), we have

$$I_{i,j} + I_{-i-2,-j-2} = I_{-1,-1}; \tag{2.7}$$

hence, $I_{-i-2,-j-2}$ is the inverse element of $I_{i,j}$.

Example 3: Letting $h = -1$ in (2.2), we find that

$$-I_{i,j} = I_{-i-2,-j-2}, \tag{2.8}$$

which is the negative element equation. Together with (2.7), we see that the inverse element is just the negative element. In particular,

$$-I_{-1,-1} = I_{-1,-1}, \tag{2.9}$$

which confirms once more that $I_{-1,-1}$ is indeed the identity element of addition. Combining (2.7) and (2.8), we have

$$I_{i,j} - I_{i,j} = I_{-1,-1}, \tag{2.10}$$

which is the equation of elimination.

Applying (2.8) to (2.1), we obtain

$$I_{i,j} = I_{i',j'} - I_{i'',j''}, \tag{2.11}$$

where $i = i' - i'' - 1$ and $j = j' - j'' - 1$. This is the subtraction formula.

Example 4: Letting $h = 0$ in (2.2), we obtain

$$0I_{i,j} = I_{-1,-1}. \tag{2.12}$$

This is the equation of zero (scalar) multiplication.

Applying (2.2) and (2.1) successively, we have $iI_{0,-1} + jI_{-1,0} = I_{i-1,-1} + I_{-1,j-1} = I_{i-1,j-1}$. Thus, $iI_{0,-1} + jI_{-1,0} = I_{-1,-1}$ if and only if $i = j = 0$. This confirms the linear independence of the two basis vectors.

Example 5: Applying (2.2), we have

$$(a+b)I_{i,j} = I_{(a+b)i+(a+b)-1,\,(a+b)j+(a+b)-1}.$$

Applying (2.2) and (2.1) successively, we have

$$\begin{aligned} aI_{i,j} + bI_{i,j} &= I_{ai+a-1,\,aj+a-1} + I_{bi+b-1,\,bj+b-1} \\ &= I_{(a+b)i+(a+b)-1,\,(a+b)j+(a+b)-1}. \end{aligned}$$

Comparing these results, we have

$$(a+b)I_{i,j} = aI_{i,j} + bI_{i,j}. \tag{2.13}$$

This is the *right distributive property* of scalar multiplication.

Again applying (2.1) and (2.2) successively, on the one hand, we have

$$\begin{aligned} h(I_{i',j'} + I_{i'',j''}) &= hI_{i'+i''+1,\,j'+j''+1} \\ &= I_{h(i'+i''+1)+h-1,\,h(j'+j''+1)+h-1}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} hI_{i',j'} + hI_{i'',j''} &= I_{hi'+h-1,\,hj'+h-1} + I_{hi''+h-1,\,hj''+h-1} \\ &= I_{h(i'+i''+1)+h-1,\,h(j'+j''+1)+h-1}. \end{aligned}$$

Comparing these results, we find that

$$h(I_{i',j'} + I_{i'',j''}) = hI_{i',j'} + hI_{i'',j''}. \tag{2.14}$$

This is the *left distributive property* of scalar multiplication.

Example 6: Let A and B denote the pair of Golden ratios, so $A + B = 1$ and $AB = -1$.

Parallel to the homogeneous case (see [4], (2.1) and (2.2)), applying (2.2), we obtain

$$I_{0,-1} + AI_{-1,0} = I_{0,-1} + I_{-1,A-1} = I_{0,A-1} \tag{2.15}$$

and

$$I_{0,-1}+BI_{-1,0}=I_{0,-1}+I_{-1,B-1}=I_{0,B-1}. \tag{2.16}$$

Subtracting (2.16) from (2.15), then applying (2.4) and (2.12), we find

$$I_{-1,0}=(I_{0,A-1}-I_{0,B-1})/(A-B), \tag{2.17}$$

which is the inhomogeneous version of Binet's formula.

This result indicates that the right-hand sides of (2.15) and (2.16) constitute the inhomogeneous Golden pair.

In terms of matrix representation, let

$$F'=\begin{bmatrix} I_{0,-1} \\ I_{-1,0} \end{bmatrix}, \quad G'=\begin{bmatrix} I_{0,B-1} \\ I_{0,A-1} \end{bmatrix}, \quad M=\begin{bmatrix} 1 & B \\ 1 & A \end{bmatrix};$$

then it can be shown that

$$MF'=G' \quad \text{and} \quad M^{-1}G'=F'. \tag{2.18}$$

Thus, these matrix relations are inhomogeneously covariant to their homogeneous counterparts (see [4], (4.8) and (4.9)).

The sum of the inhomogeneous Golden pair then gives the inhomogeneous Lucas line-sequence

$$I_{1,0}=I_{0,A-1}+I_{0,B-1}, \tag{2.19}$$

which generates the inhomogeneous Lucas line-sequence

$$L_n': \ldots -5, 2, -2, 1, 0, 2, 3, 6, \ldots, \tag{2.20}$$

where we adopt L' to represent inhomogeneous Lucas numbers, and where $L_0'=1$, $L_1'=0$ is the pair of generating numbers.

Note that (2.19) is another example of inhomogeneous covariance to its homogeneous counterpart (see [4], (3.1)). Note also that the line-sequence (2.20) is congruent to the one generated by the first Fibonacci basis vector $F_{1,0}$ with an inhomogeneous term $k=1$. The second Fibonacci basis vector $F_{0,1}$ with an inhomogeneous term $k=1$ generates the inhomogeneous line-sequence congruent to the two inhomogeneous basis vectors (1.3) and (1.4). For this reason, we are justified to refer to (1.3) and (1.4) as the inhomogeneous Fibonacci pair, as we have done above.

Furthermore, applying (2.4) to (2.19), we obtain the expression of the inhomogeneous Lucas line-sequence in terms of the inhomogeneous basis components,

$$I_{1,0}=2I_{0,-1}+I_{-1,0}. \tag{2.21}$$

This is another example of inhomogeneous covariance relating to the homogeneous relation between Lucas and Fibonacci line-sequences: $F_{2,1}=2F_{1,0}+F_{0,1}$.

3. THE TRANSLATIONAL PROPERTIES

The *translation operation* on the inhomogeneous line-sequence is defined in the same way as that on the homogeneous line-sequence (see [3], (3.2)) with the following appropriate modifications.

Definition 5: The translation operator T_i, $i =$integer, acting on a line-sequence shifts all of its elements i places to the right if $i > 0$, forming a new but congruent line-sequence

$$T_i I_{u_0,u_1} = I_{u_i,u_{i+1}}. \tag{3.1}$$

We say that translation is a congruent operation, because it preserves congruency of the line-sequence. In particular, for the additive identity, we have

$$T_i I_{-1,-1} = I_{-1,-1}. \tag{3.2}$$

Namely, the additive identity is translationally invariant.

Since translation preserves congruence, translation must be distributive over addition of line-sequences:

$$T_i(I_{u_0,u_1} + I_{u_0',u_1'}) = T_i I_{u_0,u_1} + T_i I_{u_0',u_1'}. \tag{3.3}$$

This is the *left distributive property* of translation. Using (3.1) and (2.1), we have

$$T_i(I_{u_0,u_1} + I_{u_0',u_1'}) = I_{u_i+u_i'+1,u_{i+1}+u_{i+1}'+1}. \tag{3.4}$$

Since translation preserves congruency, translation after repeated addition is the same as repeated addition after translation. Hence, multiplication and translation commute:

$$h(T_i I_{u_0,u_1}) = T_i(hI_{u_0,u_1}). \tag{3.5}$$

Definition 6: Obviously, T_i is uniquely defined; thus, two translations are said to be equal if and only if both effect the same shift of the elements in a line-sequence.

Definition 7: Addition of two translations on a line-sequence is defined to be the sum of the two translated line-sequences,

$$(T_i + T_j)I_{u_0,u_1} = T_i I_{u_0,u_1} + T_j I_{u_0,u_1}. \tag{3.6}$$

Namely, addition of translations is distributive over line-sequences. This is the *right distributive property* of translation. Obviously, addition of the translation operations is commutative,

$$T_i + T_j = T_j + T_i. \tag{3.7}$$

Applying (3.1) and (2.1) to (3.6), we obtain

$$(T_i + T_j)I_{u_0,u_1} = I_{u_i+u_j+1,\, u_{i+1}u_{j+1}+1}. \tag{3.8}$$

Therefore, the sum of two translations does not preserve the congruence of the line-sequence it operates on.

Definition 8: By the product notation, $T_i \circ T_j$, we mean successive applications of the respective translation on a line-sequence, the result of which is such that all the elements shift $i + j$ places. Hence, this is equivalent to the application of a single operation on that line-sequence. That is,

$$T_i \circ T_j = T_{i+j}. \tag{3.9}$$

Obviously, the translations commute with respect to the order of application,

$$T_i \circ T_j = T_j \circ T_i. \tag{3.10}$$

Applying (3.9) and (3.1), we have

$$(T_i \circ T_j)I_{u_0,u_1} = I_{u_{i+j},u_{i+j+1}}. \tag{3.11}$$

Letting $i = j$ and adopting the exponential convention for repeated translation, it follows from (3.9) that $T_i^2 = T_{2i}$. In general, we have

$$T_i^n = T_{ni}. \tag{3.12}$$

We illustrate the foregoing results with the following examples.

Example 7: Putting $i = j$ in (3.8), we obtain $(T_i + T_i)I_{u_0,u_1} = I_{2u_i+1,2u_{i+1}+1}$. Letting $h = 2$ in (3.5), we find that $2T_iI_{u_0,u_1} = I_{2u_i+1,2u_{i+1}+1}$. Comparing these results, we have

$$T_i + T_i = 2T_i. \tag{3.13}$$

Hence, we conclude that the scalar multiplication of a translation is equivalent to the repeated addition of that translation.

Example 8: Putting $j = i+1$ in (3.8) and applying (1.1), we get $(T_i + T_{i+1})I_{u_0,u_1} = I_{u_{i+2},u_{i+3}}$. This induces the recurrence formula of translation: $T_i + T_{i+1} = T_{i+2}$.

Let $i = 0$ and $I = T_0$, the identity of translation, then we have

$$I + T - T^2 = 0. \tag{3.14}$$

So the pleasant equation of translation (see [5], (2.16)) is inhomogeneously covariant.

Example 9: From (2.5), we have $I_{u_0,u_1} = (u_0+1)I_{I_{-1},I_0} + (u_1+1)I_{I_0,I_1}$. Applying translation on both sides and using (3.1), (3.5), and (2.1), we obtain

$$T_iI_{u_0,u_1} = I_{(u_0+1)I_{i-1}+(u_1+1)I_i+u_0+u_1+1,\,(u_0+1)I_i+(u_1+1)I_{i+1}+u_0+u_1+1}. \tag{3.15}$$

Thus, by Definition 4, the uniqueness of generating numbers, we arrive at the following formula relating u_i to the corresponding pair of I_i's:

$$u_i = (u_0+1)I_{i-1} + (u_1+1)I_i + u_o + u_1 + 1. \tag{3.16}$$

Putting $u_0 = L_0' = 1$ and $u_i = L_i' = 0$ in (3.16), we obtain the expression of the inhomogeneous Lucas numbers L_i' in terms of the inhomogeneous Fibonacci numbers:

$$L_i' = 2I_{i-1} + I_i + 2. \tag{3.17}$$

Applying (1.1), this becomes

$$L_i' = I_{i-1} + I_{i+1} + 1, \tag{3.18}$$

which is the inhomogeneous version of the relation between the Lucas numbers and the Fibonacci numbers: $L_i = F_{i-1} + F_{i+1}$.

From (3.15), we find that

$$T_iI_{1,0} = I_{2I_{i-1}+I_i+2,\,2I_i+I_{i+1}+2}. \tag{3.19}$$

Substitute (3.17) into (3.19) to obtain

$$T_iI_{L_0',L_1'} = I_{L_i',L_{i+1}'}, \tag{3.20}$$

which is none other than the translation formula for the inhomogeneous Lucas line-sequence.

Example 10: Applying (1.1) and (1.2) and using (3.1), we obtain $(T_{-1}+T_1)I_{i,0}=I_{-2,1}+I_{0,2}$.

Applying (2.1) and (2.4) to the right-hand side and using (2.12), we obtain $(T_{-1}+T_1)I_{1,0}=5I_{-1,0}$, which translates into $(T_{i-1}+T_{i+1})I_{1,0}=5T_iI_{-1,0}$.

Applying (2.20) and (1.5) to both sides, respectively, we find that

$$L'_{i-1}+L'_{i+1}=5I_i, \tag{3.21}$$

which is another relation covariant to its homogeneous counterpart $L_{i-1}+L_{i+1}=5F_i$,

Example 11: From (3.1), and putting $u_j=u'_0$ and $u_{j+1}=u'_1$, we have $T_i(T_jI_{u_o,u_1})=T_iI_{u_j,u_{j+1}}=I_{u'_i,u'_{i+1}}$, where, by (3.16), we have

$$u'_i=(u'_0+1)I_{i-1}+(u'_1+1)I_i+u'_0+u'_1+1 \quad\text{and}\quad u'_0=u_j=\ (u_0+1)I_{j-1}+(u_1+1)I_j+u_0+u_1+1.$$

On the other hand, applying (3.9) and (3.1), we obtain $T_i(T_jI_{u_0,u_1})=T_{i+j}I_{u_0,u_1}=I_{u_{i+j},u_{i+j+1}}$, where, by (3.16), we have

$$u_{i+j}=(u_0+1)I_{i+j-1}+(u_1+1)I_{i+j}+u_0+u_1+1.$$

By Definition 4, $u'_i=u_{i+j}$. Since u_0 is an independent parameter, the coefficients of u_0 must be equal in the two expressions. This leads to the following relation: $I_{j-1}I_{i-1}+I_jI_1+I_{j-1}+I_j+I_{i-1}+I_i+2=I_{i+j-1}+1$. Putting $i=j$, we obtain the relation

$$(I_{i-1}+1)^2+(I_i+1)^2=I_{2i-1}+1, \tag{3.22}$$

which is the inhomogeneous version of the relation $F_{i-1}^2+F_i^2=F_{2i-1}$. Likewise, we obtain the relation

$$(L'_i+1)(I_i+1)=I_{2i}+1, \tag{3.23}$$

which is the inhomogeneous version of the relation $L_iF_i=F_{2i}$.

Example 12: Starting from $I_{I_i,I_{i+1}}=(A-B)I_{I_i,I_{i+1}}/(A-B)$ and applying (2.13) and (2.2), we obtain the translational form of the inhomogeneous version of Binet's formula:

$$I_{I_i,I_{i+1}}=\frac{1}{A-B}\left(I_{AI_i+A-1,\,AI_{i+1}+A-1}-I_{BI_i+B-1,\,BI_{i+1}+B-1}\right). \tag{3.24}$$

Similarly, applying (2.5) to (2.19), we obtain $I_{1,0}=AI_{I_0,I_1}+BI_{I_0,I_1}+2I_{I_{-1},I_0}$. Applying translation on both sides and using (3.20) and (2.2), we get

$$I_{L'_i,L'_{i+1}}=I_{AI_i+A-1,\,AI_{i+1}+A-1}+I_{BI_i+B-1,\,BI_{i+1}+B-1}+2I_{I_{i-1},I_i}. \tag{3.25}$$

This is the translational form of Binet's formula for the inhomogeneous Lucas numbers.

4. THE INHOMOGENEOUS ANHARMONIC CASE

An anharmonic recurrence relation with an inhomogeneous constant term can be expressed in general as follows:

$$u_n=cu_{n-2}+bu_{n-1}+k, \tag{4.1}$$

where b and c, called the anharmonic parameters, are nonzero constants not both equal to one, and k is the inhomogeneous constant term. An anharmonic line-sequence is represented by

$$J_{u_0,u_1}: \ldots u_{-2}, u_{-1}, [u_0, u_1], u_2, u_3, \ldots. \tag{4.2}$$

The corresponding terms in the line-sequence are represented by

$$J_n: \ldots J_{-3}, J_{-2}, J_{-1}, J_0, J_1, J_2, J_3, \ldots, \tag{4.3}$$

where J_0 is the origin.

It is easy to see from (4.1) that anharmonic addition of anharmonic line-sequences is incompatible with the translational invariance of the additive identity. Therefore, we shall try harmonic operations as defined below.

Definition 9: Addition is inhomogeneous, that is, addition of corresponding terms in the line-sequences, together with the inhomogeneous constant k. Thus,

$$J_{i,j} = J_{i',j'} + J_{i'',j''}, \tag{4.4}$$

where $i = i' + i'' + k$ and $j = j' + j'' + k$.

Definition 10: Multiplication by a scalar h is defined in the sense of repeated addition. That is,

$$J_{i,j} = hJ_{i',j'}, \tag{4.5}$$

where h is a scalar, $i = hi' + (h-1)k$, and $j = hj' + (h-1)k$.

Definition 11: The inner product of two line-sequences is defined as follows:

$$(J_{i,j}, J_{i',j'}) = (i+k)(i'+k) + (j+k)(j'+k). \tag{4.6}$$

Two line-sequences are said to be orthogonal if and only if their inner product is zero, normal if and only if one's self inner product is one. The length of a line-sequence is defined as the (positive) square root of its inner product with itself.

Furthermore, let J_{u_0,u_1} denote the additive identity, then, for an arbitrary line-sequence $J_{i,j}$, we have $J_{u_0,u_1} + J_{i,j} = J_{i,j}$. However, by (4.4), we have $J_{u_0,u_1} + J_{i,j} = J_{u_0+i+k,\, u_1+j+k}$. Therefore, $u_0 = u_1 = -k$. So we find the additive identity

$$J_{-k,-k}: \ldots -k, -k, [-k, -k], -k, -k, \ldots. \tag{4.7}$$

On the other hand, the line-sequence of the additive identity must be translationally invariant, namely, $u_0 = u_1 = u_2$. By (4.4), $u_2 = cu_0 + bu_1 + k$, so we must have $u_0 = -k/(c+b-1)$.

Comparing these results, we arrive at the condition between the anharmonic parameters:

$$c + b = 2. \tag{4.8}$$

Then it is obvious that the set J of anharmonic line-sequences together with the inhomogeneous operations defined above constitute a vector space referred to as an *inhomogeneous-anharmonic (IA-)space*.

Let J_{u_{-1},u_0} and J_{u_0,u_1} denote the pair of basis vectors. The orthogonality requirement leads to the following combinations of basis pair choices, differing in parity: $J_{1-k,-k}$ or $J_{-1-k,-k}$ and $J_{-k,1-k}$ or $J_{-k,-1-k}$.

We choose the following combination consistent with previous works, with both the anharmonic parameters and the inhomogeneous constant specified:

$$J_{1-k,-k}(c,b;\,k)\colon \ldots, \frac{c+b^2}{c^2}-k, -\frac{b}{c}-k, [1-k,-k], c-k, cb-k; \tag{4.9}$$

$$J_{-k,1-k}(c,b;\,k)\colon \ldots, \frac{-b}{c^2}-k, \frac{1}{c}-k, [-k,1-k], b-k, c+b^2-k. \tag{4.10}$$

If $c=b=k=1$, this pair reduces to the harmonic basis pair (1.3) and (1.4) above. If $k=0$, it reduces to the homogeneous basis pair (see [3], (4.2) and (4.3)). An arbitrary line-sequence in this space can be resolved into its basis components according to the formula:

$$J_{i,j}=(i+k)J_{1-k,-k}+(j+k)J_{-k,1-k}. \tag{4.11}$$

Note that putting $h=-1$ in (4.5) gives

$$-J_{i,j}=J_{-i-2k,-j-2k}. \tag{4.12}$$

This is the negative element equation, which reduces to (2.8) if $k=1$.

Putting $h=0$ in (4.5) gives

$$0J_{i,j}=J_{-k,-k}. \tag{4.13}$$

This is the zero multiplication equation, which reduces to (2.12) if $k=1$.

Note that

$$-J_{-k,-k}=J_{-k,-k}, \tag{4.14}$$

which confirms that $J_{-k,-k}$ is indeed the additive identity.

5. THE HOMOGENEOUS ANHARMONIC CASE

It is also possible to combine line-sequences generated by (4.1), but with different inhomogeneous constant terms. To avoid confusion, we represent the set of line-sequences under these types of operations by $H_{i,j}(c,b;\,k)$. We define the following operations.

Definition 12: Addition of two line-sequences is defined as addition of corresponding terms in the line-sequences:

$$H_{i,j}(c,b;\,k)=H_{i',j'}(c,b;\,k')+H_{i'',j''}(c,b;\,k''), \tag{5.1}$$

where $i=i'+i''$, $j=j'+j''$, and $k=k'+k''$. We refer to this as *homogeneous addition*.

The additive identity is, of course, $H_{0,0}(c,b;\,0)$, namely, a sequence of zeros, and the inverse element of $H_{i,j}(c,b;\,k)$ is $H_{-i,-j}(c,b;\,-k)$.

Definition 13: Multiplication by a scalar h is defined as

$$H_{i,j}(c,b;\,k)=hH_{i',j'}(c,b;\,k'), \tag{5.2}$$

where h is scalar, $i=hi'$, $j=hj'$, and $k=hk'$. We refer to this as *homogeneous multiplication*.

Definition 14: The inner product of two line-sequences is defined as follows:

$$(H_{i,j}, H_{i',j'})=(i+k)(i'+k')+(j+k)(j'+k'). \tag{5.3}$$

Two line-sequences are said to be orthogonal if and only if their inner product is zero, normal if and only if one's self inner product is one. The length of a line-sequence is defined as the (positive) square root of its inner product with itself.

Thus, it is clear that the set H of line-sequences spans a vector space, referred to as a *homogeneous-anharmonic* (*HA-*)*space*. Obviously, the set of basis of this three-dimensional space is given by:

$$H_{1,0}(c,b;0):\ldots,\frac{c+b^2}{c^2},-\frac{b}{c},[1,0],c,cb,\ldots; \tag{5.4}$$

$$H_{0,1}(c,b;0):\ldots,\frac{-b}{c^2},\frac{1}{c},[0,1],b,c+b^2,\ldots; \tag{5.5}$$

$$H_{0,0}(c,b;1):\ldots,\frac{b-c}{c^2},\frac{-1}{c},[0,0],1,b+1,\ldots. \tag{5.6}$$

An arbitrary H line-sequence can then be decomposed into its basis components as follows:

$$H_{i,j}(c,b;k)=iH_{1,0}(c,b;0)+jH_{0,1}(c,b;0)+kH_{0,0}(c,b;1). \tag{5.7}$$

Since the operations employed in [1] are basically compatible with the homogeneous operations, many results arrived therein can be derived directly in terms of H line-sequences, but not directly in terms of I line-sequences, which undergo inhomogeneous operations. Following are some examples.

Example 13: From (5.1), we have

$$H_{0,0}(1,1;1)=H_{1,1}(1,1;0)+H_{-1,-1}(1,1;1), \tag{5.8}$$

which corresponds to (1.4) in [1];

$$H_{1,2}(1,1;1)=H_{1,2}(1,1;0)+H_{0,0}(1,1;1), \tag{5.9}$$

which corresponds to (1.14) in [1];

$$H_{2,1}(1,1;1)=H_{2,1}(1,1;0)+H_{0,0}(1,1;1), \tag{5.10}$$

which corresponds to (1.22) in [1]; and so forth.

Example 14: Substituting $c=b=1$ into (5.7), we obtain

$$H_{i,j}(1,1;k)=iH_{1,0}(1,1;0)+jH_{0,1}(1,1;0)+kH_{0,0}(1,1;1), \tag{5.11}$$

which corresponds to (2.2) in [1], or to (1.13) in [1] if $k=1$.

Substituting (5.8) into (5.11) and using (5.2) and the distributive property of multiplication, we obtain

$$H_{i,j}(1,1;k)=iH_{1,0}(1,1;0)+jH_{0,1}(1,1;0)+kH_{1,1}(1,1;0)+H_{-k,-k}(1,1;k), \tag{5.12}$$

which corresponds to (2.3) in [1], or to (1.6) in [1] if $k=1$.

Using (5.1) and (5.2), we have

$$H_{0,0}(1,1;k)=H_{1,2}(1,1;k)-H_{1,0}(1,1;0)-2H_{0,1}(1,1;0).$$

Substituting into (5.11), we obtain

$$H_{i,j}(1,1;k) = (i-k)H_{1,0}(1,1;0) + (j-2k)H_{0,1}(1,1;0) + kH_{1,2}(1,1;1), \tag{5.13}$$

which corresponds to (2.6) in [1]. It reduces to (1.33) in [1] if $k = 1$. And so forth.

Following is a table of some equivalence and correspondence ($\to$) relations.

TABLE 1. Some Equivalence and Correspondence Relations

No.	Relations	References
1	$J_{i,j}(1,1;0) = F_{i,j}$	[3], (1.3)
2	$J_{i,j}(c,b;0) = G_{i,j}$	[3], (4.1)
3	$J_{i,j}(1,1;1) = I_{i,j}$	(1.1)
4	$H_{1,1}(1,1;0) \to F_n$	[1], p. 193
5	$H_{1,1}(1,1;1) \to c_n$	[1], (1.2)
6	$H_{1,2}(1,1;1) \to c_n^*$	[1], (1.3)
7	$H_{0,0}(1,1;1) \to c_n'$	[1], (1.4)
8	$H_{a,b}(1,1;1) \to c_n(a,b)$	[1], (1.5)

REFERENCES

1. Marjorie Bicknell-Johnson & Gerald E. Bergum. "The Generalized Fibonacci Numbers $\{c_n\}$, $c_n = c_{n-1} + c_{n-2} + K$." In *Applications of Fibonacci Numbers* **3:**193-205. Dordrecht: Kluwer, 1988.
2. Ray M. Bowen & C. C. Wang. *Introduction to Vectors and Tensors*, pp. 40-59. New York: Plenum Press, 1976.
3. J. Y. Lee. "Some Basic Properties of the Fibonacci Line-Sequence." In *Applications of Fibonacci Numbers* **4:**203-14. Dordrecht: Kluwer, 1990.
4. J. Y. Lee. "The Golden-Fibonacci Equivalence." *The Fibonacci Quarterly* **30.3** (1992):216-20.
5. J. Y. Lee. "Some Basic Translational Properties of the General Fibonacci Line-Sequence." In *Applications of Fibonacci Numbers* **6:**339-74. Dordrecht: Kluwer, 1994.
6. Bolian Liu. "A Matrix Method To Solve Linear Recurrences with Constant Coefficients." *The Fibonacci Quarterly* **30.1** (1992):2-8.

AMS Classification Numbers: 11B39, 15A03

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