ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-831 ***Proposed by the editor***

Find a polynomial $f(x, y)$ with integer coefficients such that $f(F_n, L_n) = 0$ for all integers n.

B-832 ***Proposed by Andrew Cusumano, Great Neck, NY***

Find a pattern in the following numerical identities and create a formula expressing a more general result.

$$\begin{aligned}
3^5+2^5+1^5+1^5 &= 5\cdot 3^4 - 128\\
5^5+3^5+2^5+1^5+1^5 &= 8\cdot 5^4 - 128 - 2\cdot 3\cdot 5(19+2\cdot 3\cdot 5)\\
8^5+5^5+3^5+2^5+1^5+1^5 &= 13\cdot 8^4 - 128 - 2\cdot 3\cdot 5(19+2\cdot 3\cdot 5)\\
&\quad - 3\cdot 5\cdot 8(19+2\cdot 3\cdot 5+2\cdot 5\cdot 8)\\
13^5+8^5+5^5+3^5+2^5+1^5+1^5 &= 21\cdot 13^4 - 128 - 2\cdot 3\cdot 5(19+2\cdot 3\cdot 5)\\
&\quad - 3\cdot 5\cdot 8(19+2\cdot 3\cdot 5+2\cdot 5\cdot 8)\\
&\quad - 5\cdot 8\cdot 13(19+2\cdot 3\cdot 5+2\cdot 5\cdot 8+2\cdot 8\cdot 13)\\
21^5+13^5+8^5+5^5+3^5+2^5+1^5+1^5 &= 34\cdot 21^4 - 128 - 2\cdot 3\cdot 5(19+2\cdot 3\cdot 5)\\
&\quad - 3\cdot 5\cdot 8(19+2\cdot 3\cdot 5+2\cdot 5\cdot 8)\\
&\quad - 5\cdot 8\cdot 13(19+2\cdot 3\cdot 5+2\cdot 5\cdot 8+2\cdot 8\cdot 13)\\
&\quad - 8\cdot 13\cdot 21(19+2\cdot 3\cdot 5+2\cdot 5\cdot 8+2\cdot 8\cdot 13+2\cdot 13\cdot 21)
\end{aligned}$$

B-833 ***Proposed by Al Dorp, Edgemere, NY***

For n a positive integer, let $f(x)$ be the polynomial of degree $n-1$ such that $f(k) = L_k$ for $k = 1, 2, 3, \ldots, n$. Find $f(n+1)$.

B-834 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia***

For x a real number and n an integer larger than 1, prove that

$$(x+1)F_1+(x+2)F_2+\cdots+(x+n)F_n<2^n\sqrt{\frac{n(n+1)(2n+1+6x)+nx^2}{6}}.$$

B-835 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY***

In a sequence of coin tosses, a *single* is a term (H or T) that is not the same as any adjacent term. (For example, in the sequence HHTHHHTH, the singles are the terms in positions 3, 7, and 8.) Let $S(n,r)$ be the number of sequences of n coin tosses that contain exactly r singles. If $n\ge 0$ and p is a prime, find the value modulo p of $\frac{1}{2}S(n+p-1,p-1)$.

NOTE: The Elementary Problems column is in need of more *easy*, yet elegant and nonroutine problems.

SOLUTIONS

Perfect Squares

B-814 ***(Corrected) Proposed by M. N. Deshpande, Institute of Science, Nagpur, India (Vol. 34, no. 4, August 1996)***

Show that, for each positive n, there exists a constant C_n such that $F_{2n+2i}F_{2i}+C_n$ and $F_{2n+2i+1}F_{2i+1}-C_n$ are both perfect squares for all positive integers i.

Solution by Paul S. Bruckman, Seattle, WA

It is easy to show (for example, by using the Binet forms) that

$$F_{2n+j}F_j+(-1)^jF_n^2=F_{n+j}^2$$

holds for all integers n and j. Thus, $C_n=F_n$ is the solution.

Also solved by Brian D. Beasley, Russell Euler and Jawad Sadek, Herta T. Freitag, Hans Kappus, Harris Kwong, Carl Libis, David E. Manes, Bob Prielipp, H.-J. Seiffert, I. Strazdins, and the proposer.

Ternary Cubic Forms

B-815 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 34, no. 4, August 1996)***

Let $K(a,b,c)=a^3+b^3+c^3-3abc$. Show that, if x_1, x_2, x_3, y_1, y_2, and y_3 are integers, then there exist integers z_1, z_2, and z_3 such that

$$K(x_1,x_2,x_3)\cdot K(y_1,y_2,y_3)=K(z_1,z_2,z_3).$$

Solution by H.-J. Seiffert, Berlin, Germany

Let

$$M(a,b,c)=\begin{pmatrix} a & b & c\\ c & a & b\\ b & c & a\end{pmatrix}.$$

A straightforward but tedious calculation shows that

$$M(x_1, x_2, x_3)\cdot M(y_1, y_2, y_3) = M(z_1, z_2, z_3),$$

where

$$\begin{aligned} z_1 &= x_1y_1 + x_2y_3 + x_3y_2, \\ z_2 &= x_1y_2 + x_2y_1 + x_3y_3, \\ z_3 &= x_1y_3 + x_2y_2 + x_3y_1. \end{aligned}$$

It is easily verified that $\det M(a,b,c) = K(a,b,c)$. Thus, $K(x_1, x_2, x_3)\cdot K(y_1, y_2, y_3) = K(z_1, z_2, z_3)$, where z_1, z_2, and z_3 are as given above.

Clary found this result in [1].

Reference

1. G. Chrystal. *Textbook of Algebra*, Part I, exercise 21, p. 84. New York: Dover Publications, 1961.

Also solved by Brian D. Beasley, Stuart Clary, Hans Kappus, Bob Prielipp, Adam Stinchcombe, David C. Terr, and the proposer.

Triple Rational Inequality

B-816 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN (Vol. 34, no. 4, August 1996)***

Let i, j, and k be any three positive integers. Show that

$$\frac{F_jF_k}{F_i + F_iF_jF_k} + \frac{F_kF_i}{F_j + F_iF_jF_k} + \frac{F_iF_j}{F_k + F_iF_jF_k} < 2.$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

We prove the following generalization: Given $n \ge 3$, let $x_1, x_2, \ldots, x_n$ be positive integers and let $x = x_1x_2\cdots x_n$. Then

$$S = \sum_{i=1}^{n} \frac{x/x_i}{x_i + x} < n - 1.$$

We begin the proof by assuming, without loss of generality, that $x_1 \le x_2 \le \cdots \le x_n$. If $x_{n-1} \ge 2$, then

$$\begin{aligned} S &= \frac{x/x_1}{x_1 + x} + \frac{x/x_2}{x_2 + x} + \cdots + \frac{x/x_n}{x_n + x} < \frac{x/x_1}{x} + \frac{x/x_2}{x} + \cdots + \frac{x/x_n}{x} \\ &= \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \le (n-2)(1) + 1/2 + 1/2 = n - 1. \end{aligned}$$

If $x_{n-1} < 2$, then $x_1 = x_2 = \cdots = x_{n-1} = 1$, so $x = x_n$. Thus,

$$S = (n-1)\frac{x_n}{1+x_n} + \frac{1}{2x_n} = \frac{2(n-1)x_n^2 + x_n + 1}{2x_n(1+x_n)} < \frac{2(n-1)x_n^2 + 2(n-1)x_n}{2x_n^2 + 2x_n} = n - 1$$

since $n \ge 3$ implies that $(2n-3)x_n > 1$.

The result in Problem B-816 now follows by taking $n = 3$ and $x_1 + 1 = F_i$, $x_2 = F_j$, and $x_3 = F_k$.

Also solved by Michel A. Ballieu, Paul S. Bruckman, H.-J. Seiffert, Adam Stinchcombe, and the proposer.

Radical Integer

B-817 ***Proposed by Kung-Wei Yang, Western Michigan University, Kalamazoo, MI (Vol. 34, no. 4, August 1996)***

Show that

$$\sqrt[k]{\sum_{i=0}^{k}\binom{k}{i}F_{ni-1}F_{n(k-i)+1}-\sum_{j=1}^{k-1}\binom{k}{j}F_{nj}F_{n(k-j)}}$$

is an integer for all positive integers k and n.

Solution by Paul S. Bruckman, Seattle, WA

We use the identity $F_{u-1}F_{v+1}-F_uF_v=(-1)^vF_{u-v-1}$. Set $u=ni$ and $v=n(k-i)$ and note that the second sum under the radical in the statement of the problem may include the terms $j=0$ and $j=k$. Replacing j by i, the given expression under the radical sign is transformed as follows:

$$\begin{aligned}\sum_{i=0}^{k}\binom{k}{i}(-1)^{n(k-i)}F_{n(2i-k)-1}&=5^{-1/2}\sum_{i=0}^{k}\binom{k}{i}(-1)^{n(k-i)}[\alpha^{n(2i-k)-1}-\beta^{n(2i-k)-1}]\\&=5^{-1/2}[\alpha^{-nk-1}(\alpha^{2n}+(-1)^n)^k-\beta^{-nk-1}(\beta^{2n}+(-1)^n)^k]\\&=5^{-1/2}[-\beta(\alpha^n+\beta^n)^k+\alpha(\alpha^n+\beta^n)^k]\\&=(\alpha^n+\beta^n)^k=L_n^k.\end{aligned}$$

Therefore, the given expression reduces to L_n, which is, of course, an integer (independent of k).

Also solved by H.-J. Seiffert, David Zeitlin, and the proposer.

Binomial Harmonic Sum

B-818 ***Proposed by L. C. Hsu, Dalian University of Technology, Dalian, China (Vol. 34, no. 4, August 1996)***

Let $H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Find a closed form for

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{2k}.$$

Editorial Note: *This problem also appeared as Problem 60 in the November 1996 issue of the journal* Math Horizons. *I apologize for the duplication. The problem column editor for* Math Horizons *asked me if I had any problems they could use, and since this problem (which was originally submitted here) did not involve Fibonacci numbers, I released it to them with the author's permission. Unfortunately, I forgot to delete it from my computer files, so when we started running low on problems, I inadvertently used it.*

Solution by Hans Kappus, Rodersdorf, Switzerland

Let us tackle the more general sum

$$S(p,n)=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{pk},$$

where p and n are positive integers.

In a first attempt, one may try to look for a recurrence. Thus,

$$S(p,n+1)=\sum_{k=1}^{n}(-1)^{k-1}\left[\binom{n}{k}+\binom{n}{k-1}\right]H_{pk}+(-1)^{n}H_{p(n+1)}$$
$$=S(p,n)+H_p+\sum_{k=1}^{n}(-1)^{k}\binom{n}{k}H_{p(k+1)}.$$

But

$$H_{p(k+1)}=H_p+\sum_{i=1}^{p}\frac{1}{pk+i}.$$

Hence,

$$S(p,n+1)=\sum_{i=1}^{p}J_p(i,n),$$

where

$$J_p(i,n)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{1}{pk+i}.$$

Using

$$\frac{1}{pk+i}=\int_0^1 x^{pk+i-1}\,dx$$

and the Binomial Theorem leads to

$$J_p(i,n)=\int_0^1 x^{i-1}(1-x^p)^n\,dx;\quad 1\le i\le p.$$

Obviously, $J_p(i,0)=1/i$ and, using integration by parts, we find

$$J_p(i,n)=\frac{pn}{i}J_p(p+i,n-1).$$

The explicit formula

$$J_p(i,n)=p^n n!\prod_{j=0}^{n}\frac{1}{jp+i}$$

can now be proved by an easy induction. Writing n instead of $n+1$ again, our final result reads:

$$S(p,n)=p^{n-1}(n-1)!\sum_{i=1}^{p}\prod_{j=0}^{n-1}\frac{1}{jp+i}.$$

For the special case $p=2$, we get the neat formula

$$S(2,n)=\frac{1}{2n}+\frac{2^{2n-1}}{n\binom{2n}{n}}.$$

Lord found the related formula: $\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k=\frac{1}{n}$.

Also solved by Paul S. Bruckman, Carl Libis, Graham Lord, David E. Manes, H.-J. Seiffert, David Zeitlin, and the proposer.

Finding a Pellian Identity

B-819 ***Proposed by David Zeitlin, Minneapolis, MN***
(Vol. 34, no. 4, August 1996)

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$P_n = P_{n-a} + 444P_{n-b} + P_{n-c} + P_{n-d},$$

where P_n is the Pell sequence, defined by $P_{n+2} = 2P_{n+1} + P_n$, for $n \geq 0$, with $P_0 = 0$, $P_1 = 1$.

Solution by H.-J. Seiffert, Berlin, Germany

If Q_n is the Pell-Lucas sequence, defined by the recurrence $Q_{n+2} = 2Q_{n+1} + Q_n$, with initial conditions $Q_0 = Q_1 = 2$, then (see [1], page 12, equations 3.22 and 3.24)

$$Q_r P_m = P_{m+r} + (-1)^r P_{m-r}$$

for all integers r and m. From this equation, it easily follows that

$$P_n = P_{n-u+v} + (Q_u - Q_v)P_{n-u} + P_{n-u-v} + P_{n-2u}$$

if u is odd and v is even.

Taking $u = 7$ and $v = 4$, and noting that $Q_4 = 34$ and $Q_7 = 478$, we find

$$P_n = P_{n-3} + 444P_{n-7} + P_{n-11} + P_{n-14},$$

showing that $a = 3$, $b = 7$, $c = 11$, and $d = 14$ work.

Reference

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.

Also solved by Paul S. Bruckman, Curtis Cooper, Daina Krigens, Carl Libis, David E. Manes, and the proposer.

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