SUMMATION FORMULAS FOR SPECIAL LEHMER NUMBERS

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1. INTRODUCTION AND PRELIMINARIES

The observation made in [2] brings to the attention of the reader the fact that an improper use of the geometric series formula (g.s.f.) for obtaining summation formulas for the well-known generalized sequences $\{W_n(0,1;p,q)\}$ and $\{W_n(2,p;p,q)\}$ (e.g., see [5]) leads to meaningless expressions when p and q assume certain special values. The same problem may arise if we seek summation formulas for *Lehmer numbers* (e.g., see [4] and [7] for recent studies on the properties of these numbers).

The closed-form expression (Binet form) for the n^{th} Lehmer number $U_n(p,q)$ (or simply U_n is no misunderstanding can arise) is

$$U_n = \begin{cases} (\alpha^n - \beta^n)/(\alpha^2 - \beta^2) & (n \text{ even}), \\ (\alpha^n - \beta^n)/(\alpha - \beta) & (n \text{ odd}), \end{cases} \tag{1.1}$$

α and β being the roots of the equation $x^2 - \sqrt{p}x + q = 0$, where p and q are integers. These roots are given by

$$\begin{cases} \alpha = (\sqrt{p} + \sqrt{p-4q})/2 = (\sqrt{p} + \Delta)/2, \\ \beta = (\sqrt{p} - \Delta)/2, \end{cases} \tag{1.2}$$

so that

$$\alpha + \beta = \sqrt{p}, \tag{1.3}$$

$$\alpha - \beta = \Delta, \tag{1.4}$$

$$\alpha\beta = q, \tag{1.5}$$

$$\alpha^2 + \beta^2 = (p + \Delta^2)/2 = p - 2q. \tag{1.6}$$

The numbers U_n obey (e.g., see [7]) the second-order recurrence relations

$$U_n = \begin{cases} U_{n-1} - qU_{n-2} & (n \geq 2 \text{ even}), \\ pU_{n-1} - qU_{n-2} & (n \geq 3 \text{ odd}), \end{cases} \tag{1.7}$$

with initial conditions

$$U_0 = 0 \quad \text{and} \quad U_1 = 1, \tag{1.7$'$}$$

whence it can be observed that $U_n(1,-1)$ is the n^{th} Fibonacci number. As done in [7], without loss of generality, we can assume that

$$p > 0, \tag{1.8}$$

$$p - 4q > 0, \tag{1.9}$$

$$q \neq 0. \tag{1.10}$$

The simplest summation formula for *normal* Lehmer numbers [that is, Lehmer numbers with arbitrary parameters p and q satisfying (1.8)-(1.10)] is

$$\sum_{n=0}^{N} U_n = \frac{q^2(U_N + U_{N-1}) - U_{N+1} - U_{N+2} + q + 2}{q^2 + 2q - p + 1}. \tag{1.11}$$

This formula can be obtained after some manipulation involving the use of (1.1) [cf. (2.6) and (2.7)], (1.5), (1.6), and the g.s.f. The relation $U_2 = U_1 = 1$ [see (1.7) and (1.7')] must also be used.

One can immediately observe that (1.11) *does not have general validity*. In fact, if $p = k^2$ (k a positive integer) and

$$q = -1 \pm k, \tag{1.12}$$

then the denominator on the right-hand side of (1.11) vanishes. The same problem arises also in *general* summation formulas (that is, summations where the subscripts of the summands are in arithmetical progression with arbitrary difference) for normal Lehmer numbers.

The principal aim of this note is to establish general summation formulas for a subset of normal Lehmer numbers: the *special* Lehmer numbers $U_n(k^2, -1 \pm k)$. As a concluding remark, some simple properties of these numbers are pointed out in Section 5. To save space, the number of proofs has been kept to a minimum.

2. SUMMATION FORMULAS FOR NORMAL LEHMER NUMBERS: BASIC RELATIONS

For notational convenience, let us define

$$S_N(h, r) = \sum_{n=0}^{N} (\alpha^{hn+r} - \beta^{hn+r}), \tag{2.1}$$

$$\delta = 1/(\alpha - \beta), \tag{2.2}$$

$$\gamma = 1/(\alpha^2 - \beta^2). \tag{2.3}$$

The following relations are fundamental tools for establishing general summation formulas for normal Lehmer numbers. They can be proved readily by simply using (1.1) and will be used to obtain summation formulas for special Lehmer numbers.

$$\sum_{n=0}^{N} U_{hn+r} = \gamma S_N(h, r) \quad (h \text{ and } r \text{ even, } N \text{ arbitrary}), \tag{2.4}$$

$$= \delta S_N(h, r) \quad (h \text{ even, } r \text{ odd, } N \text{ arbitrary}), \tag{2.5}$$

$$= \gamma S_{N/2}(2h, r) + \delta S_{N/2-1}(2h, h+r) \quad (h \text{ odd, } r \text{ and } N \text{ even}), \tag{2.6}$$

$$= \gamma S_{(N-1)/2}(2h, r) + \delta S_{(N-1)/2}(2h, h+r) \quad (h \text{ and } N \text{ odd, } r \text{ even}), \tag{2.7}$$

$$= \gamma S_{N/2-1}(2h, h+r) + \delta S_{N/2}(2h, r) \quad (h \text{ and } r \text{ odd, } N \text{ even}), \tag{2.8}$$

$$= \gamma S_{(N-1)/2}(2h, h+r) + \delta S_{(N-1)/2}(2h, r) \quad (h, r, \text{ and } N \text{ odd}). \tag{2.9}$$

3. SUMMATION FORMULAS FOR $U_n(k^2, -1 \pm k)$

3.1 Summation Formulas for $U_n(k^2, k-1)$

If $p = k^2$ and

$$q = k - 1, \tag{3.1}$$

then conditions (1.8)-(1.10) imply that we must have

$$k^2 \geq 9 \tag{3.2}$$

which, by (3.1), implies that

$$q \geq 2. \tag{3.3}$$

If (3.1) holds, then from (1.2) we have

$$\begin{cases} \alpha = k - 1 = q, \\ \beta = 1. \end{cases} \tag{3.4}$$

By using (3.4), (2.4)-(2.9) and the g.s.f. *properly* (that is, taking the value of β into account), we get the following general summation formulas:

(i) For $h \geq 2$ even

$$\sum_{n=0}^{N} U_{hn+r} = \frac{1}{q^2-1}\left[\frac{U_{h(N+1)+r} - U_r}{U_h} - (N+1)\left(1 + q\frac{1-(-1)^r}{2}\right)\right]. \tag{3.5}$$

(ii) For h odd

$$\sum_{n=0}^{N} U_{hn+r} = \frac{1}{q^2-1}\left[\frac{U_{h(N+2)+r} - U_{h(N+1)+r} - U_{h+r} - U_r}{U_{2h}} - \frac{2(q+2)(N+1) - q(-1)^r[1+(-1)^N]}{4}\right]. \tag{3.6}$$

The proofs of (3.5) and (3.6) are easy but rather tedious. For the sake of brevity, only the proof of (3.6) (for r odd and N even) will be given. Observe that letting $h = 1$ and $r = 0$ in (3.6) yields the identity

$$\sum_{n=0}^{N} U_n = \frac{1}{q^2-1}\left[U_{N+2} + U_{N+1} - \frac{2[4+(q+2)N] + [1-(-1)^N]q}{4}\right], \tag{3.7}$$

which gives the correct closed-form expression for the left-hand side of (1.11) in this case. We do not exclude the possibility that more compact expressions for (3.5) and (3.6) can be found.

Proof of (3.6) (for r odd and N even): First, use (2.8), (2.1)-(2.3), and (3.4), along with the g.s.f. to write

$$\begin{aligned} \sum_{n=0}^{N} U_{hn+r} &= \frac{1}{q^2-1}\sum_{n=0}^{N/2-1}(q^{(2n+1)h+r} - 1) + \frac{1}{q-1}\sum_{n=0}^{N/2}(q^{2nh+r} - 1) \\ &= \frac{1}{q^2-1}\left(q^{h+r}\frac{q^{hN}-1}{q^{2h}-1} - \frac{N}{2}\right) + \frac{1}{q-1}\left(q^r\frac{q^{h(N+2)}-1}{q^{2h}-1} - \frac{N+2}{2}\right). \end{aligned} \tag{3.8}$$

Then, take into account the parity of N and r, and use (1.1) and (3.4) to rewrite (3.8) as

$$\sum_{n=0}^{N} U_{nh+r} = \frac{U_{h(N+2)+r} + U_{h(N+1)+r} - U_{h+r} - U_r}{q^{2h}-1} - \frac{N}{2(q^2-1)} - \frac{(q+1)(N+2)}{2(q+1)(q-1)}$$

$$= \frac{U_{h(N+2)+r} + U_{h(N+1)+r} - U_{h+r} - U_r}{(q^2-1)U_{2h}} - \frac{qN+2q+2(N+1)}{2(q^2-1)}$$

$$= idem - \frac{(q+2)(N+1)+q}{2(q^2-1)}. \quad \text{Q.E.D.}$$

3.2 Summation Formulas for $U_n(k^2, -k-1)$

If $p = k^2$ and

$$q = -k-1, \tag{3.9}$$

then conditions (1.8)-(1.10) imply that we must have

$$k^2 \geq 1 \tag{3.10}$$

which, by (3.9), implies that

$$q \leq -2. \tag{3.11}$$

If (3.9) holds, then from (1.2) we have

$$\begin{cases} \alpha = k+1 = -q, \\ \beta = -1. \end{cases} \tag{3.12}$$

Remark 1: By replacing (3.4) and (3.12) in (1.1), one can observe that $U_n(k^2, -1+k)$ and $U_n(k^2, -1-k)$ have the same form as functions of q. Hence, we obtain summation formulas that are identical to (3.5) and (3.6).

4. OTHER SUMMATION FORMULAS

Of course, other kinds of summation formulas for $U_n(k^2, -1 \pm k)$ may be of interest. As a minor example, we show the following closed-form expressions:

$$\sum_{n=0}^{N} (-1)^n U_n = \frac{1}{q^2-1}\left[(-1)^N (U_{N+2} - U_{N+1}) + \frac{2qN + (q+2)[1-(-1)^N]}{4}\right], \tag{4.1}$$

$$\sum_{n=0}^{N} \binom{N}{n} U_n = \frac{(q+2)[(q+1)^N - 2^N] - q(1-q)^N}{2(q^2-1)}, \tag{4.2}$$

and

$$\sum_{n=0}^{N} nU_n = \begin{cases} \dfrac{N(A_{N+4} - A_{N+2}) - 2q^2(A_{N+1} - 1) + (q^2-1)(U_{N+1} - 1)}{(q^2-1)^2} - X_N & (N \text{ even}), \\[2ex] \dfrac{(N-1)(A_{N+4} - A_{N+2}) - 2q^2(A_N - 1) + (q^2-1)(U_{N+2} - 1)}{(q^2-1)^2} - Y_N & (N \text{ odd}), \end{cases} \tag{4.3}$$

where

$$A_N = U_N + U_{N-1}, \tag{4.4}$$

$$X_n = N[(q+2)N+2]/[4(q^2-1)], \tag{4.5}$$

$$Y_N = (N+1)[(q+2)N+q]/[4(q^2-1)]. \tag{4.6}$$

The (partial) proof of (4.2) is given below, whereas the proofs of (4.1) and (4.3) are left as an exercise for the interested reader. We confine ourselves to mentioning that the proof of (4.3) involves the use of (3.1) of [1].

It has to be noted that the summation formulas (3.5), (3.6), and (4.1)-(4.3) also apply for negative values of h and/or r. Obviously, the extension of (normal) Lehmer numbers through negative values of the subscripts may be required. In fact, from (1.1) and (1.5), we readily get

$$U_{-n} = -U_n / q^n. \tag{4.7}$$

***Proof of (4.2)* [$U_n \equiv U_n(k^2, k-1)$, N even]:** By using (2.6), (3.4), and the identities available in [6; Ex. 4, p. 133], the left-hand side of (4.2) can be written as

$$\frac{1}{q^2-1}\sum_{n=0}^{N/2}\binom{N}{2n}q^{2n} + \frac{1}{q-1}\sum_{n=0}^{N/2-1}\binom{N}{2n+1}q^{2n+1} - \frac{1}{q^2-1}\left[\sum_{n=0}^{N/2}\binom{N}{2n} + (q+1)\sum_{n=0}^{N/2-1}\binom{N}{2n+1}\right]$$

$$= \frac{1}{2(q^2-1)}[(1+q)^N + (1-q)^N] + \frac{1}{2(q-1)}[(1+q)^N - (1-q)^N] - \frac{2^N + q2^{N-1}}{q^2-1}$$

$$= \frac{(q+2)[(1+q)^N - 2^N] - q(1-q)^N}{2(q^2-1)}. \quad \text{Q.E.D.}$$

Remark 2: By virtue of Remark 1, the case $U_n \equiv U_n(k^2, -k-1)$ (N even) is also covered by the above proof.

5. CONCLUDING REMARKS

Let us conclude this note by pointing out some simple properties of the special Lehmer numbers that might be of some interest. For notational convenience, put

$$U_n(k^2, k-1) \overset{\text{def}}{=} U_n^+(k) = \begin{cases} [(k-1)^n - 1]/[k(k-2)] & (n \text{ even}) \\ [(k-1)^n - 1]/(k-2) & (n \text{ odd}) \end{cases} \quad (k \geq 3), \tag{5.1}$$

$$U_n(k^2, -k-1) \overset{\text{def}}{=} U_n^-(k) = \begin{cases} [(k+1)^n - 1]/[k(k+2)] & (n \text{ even}) \\ [(k+1)^n + 1]/(k+2) & (n \text{ odd}) \end{cases} \quad (k \geq 1). \tag{5.2}$$

(i) From (5.1) and (5.2), the following identities can easily be derived.

$$U_{2n}^-(k) = U_{2n}^+(k+2), \tag{5.3}$$

$$U_{2n+1}^-(k) = [kU_{2n+1}^+(k+2) + 2]/(k+2), \tag{5.4}$$

$$U_n^+(k) = [U_{n+2}^+(k) + (k-1)^2 U_{n-2}^+(k)]/[k(k-2)+2], \tag{5.5}$$

$$U_n^-(k) = [U_{n+2}^-(k) + (k+1)^2 U_{n-2}^-(k)]/[k(k+2)+2], \tag{5.6}$$

$$U_{2n}^{+}(k) = [U_{2n+1}^{+}(k) - U_{2n+2}^{+}(k)]/(k-1) \tag{5.7}$$

$$= -1 + [U_{2n+1}^{+}(k) - (k-1)(k-2)U_{2n-1}^{+}(k) + 1]/k, \tag{5.7'}$$

$$U_{2n}^{-}(k) = [U_{2n+2}^{-}(k) - U_{2n+1}^{-}(k)]/(k+1) \tag{5.8}$$

$$= [U_{2n+1}^{-}(k) - (k+1)U_{2n-1}^{-}(k)]/k^2. \tag{5.8'}$$

(ii) As a final remark, we observed that the numbers $U_n^+(k)$ seem to be related to the *central factorial numbers of the second kind* $T(h, n)$ (e.g., see [3]). More precisely, we found the following identities, the proofs of which are based on (1.1), (3.4), (4.7), and the definition of $T(h, n)$.

Proposition: For n an arbitrary integer, we have

$$U_{2n}^{+}(3) = T(2n+2, 4), \tag{5.9}$$

$$U_{2n}^{+}(4) = 4^{n-1}T(2n+1, 3), \tag{5.10}$$

$$U_{2n+1}^{+}(5) = T(4n+4, 4). \tag{5.11}$$

A possible generalization of (5.9)-(5.11) will be the object of a future study.

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