CONICS WHICH CHARACTERIZE CERTAIN LUCAS SEQUENCES

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1. INTRODUCTION

In the notation of Horadam [3], write

$$W_n = W_n(a, b; p, q), \tag{1.1}$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b, \quad n \geq 2. \tag{1.2}$$

The sequence $\{W_n\}_{n=0}^{\infty}$ can be extended to negative subscripts using (1.2); we write simply $\{W_n\}$.

We shall be concerned with the sequences

$$\begin{cases} U_n = W_n(0, 1; P, -1), \\ V_n = W_n(2, P; P, -1), \end{cases} \tag{1.3}$$

where $P \neq 0$ is an integer, and

$$\begin{cases} u_n = W_n(0, 1; p, 1), \\ v_n = W_n(2, p; p, 1), \end{cases} \tag{1.4}$$

where $|p| > 2$ is also an integer.

For the sequences (1.3) and (1.4), we define $\Delta = P^2 + 4$ and $D = p^2 - 4$, respectively. Taking α and β to be the roots of $x^2 - Px - 1 = 0$, we have the well-known expressions (the Binet forms)

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n. \tag{1.5}$$

Similarly, if γ and δ are the roots of $x^2 - px + 1 = 0$, then

$$u_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad v_n = \gamma^n + \delta^n. \tag{1.6}$$

According to Dickson ([2], p. 405), Lucas proved that if x and y are consecutive Fibonacci numbers, then (x, y) is a lattice point on one of the hyperbolas

$$y^2 - xy - x^2 = \pm 1, \tag{1.7}$$

and Wasteels proved the converse in 1902. Interest in conics whose equations are satisfied by pairs of successive terms of linear recursive sequences has been rekindled. See, for example, [1], [4], and [5]. Recently McDaniel [6] has provided converses to several of the results of these writers. For example, he proved the following.

Theorem: Let x and y be positive integers. The pair (x, y) is a solution of $y^2 - Pxy - x^2 = \pm 1$ if and only if there exists a positive integer n such that $x = U_n$ and $y = U_{n+1}$.

The object of this paper is to generalize McDaniel's results and to obtain new ones.

2. SOME PRELIMINARY RESULTS

Throughout this paper, m and n denote integers. Also, Δ and D are as defined in Section 1. For the sequences (1.3), we record the following results, each of which can be proved using the Binet forms:

$$V_m^2 - 4 = \Delta U_m^2, \quad m \text{ even}, \tag{2.1}$$

$$V_m^2 + 4 = \Delta U_m^2, \quad m \text{ odd}, \tag{2.2}$$

$$U_n V_m + V_n U_m = 2U_{n+m}, \tag{2.3}$$

$$U_n V_m - V_n U_m = \begin{cases} 2U_{n-m}, & m \text{ even}, \\ -2U_{n-m}, & m \text{ odd}, \end{cases} \tag{2.4}$$

$$V_n V_m + \Delta U_n U_m = 2V_{n+m}, \tag{2.5}$$

$$V_n V_m - \Delta U_n U_m = \begin{cases} 2V_{n-m}, & m \text{ even}, \\ -2V_{n-m}, & m \text{ odd}. \end{cases} \tag{2.6}$$

We shall also need the following results:

Lemma 1: The integer solutions of $\Delta x^2 + 4 = z^2$ are precisely the pairs $(x, z) = (\pm U_{2n}, \pm V_{2n})$.

Lemma 2: The integer solutions of $\Delta x^2 - 4 = z^2$ are precisely the pairs $(x, z) = (\pm U_{2n+1}, \pm V_{2n+1})$.

These two lemmas constitute the first half of McDaniel's Corollary 1, a well-known result for which he provides an alternative proof.

Lemma 3: If Δ is square free, the integer solutions of $\Delta(x^2 - 4) = z^2$ are precisely the pairs $(x, z) = (\pm V_{2n}, \pm \Delta U_{2n})$.

Proof: Since Δ is square free and $\Delta | z^2$, then $\Delta | z$. Writing $z = \Delta z_0$ we obtain $\Delta z_0^2 + 4 = x^2$, and the use of Lemma 1 completes the proof. □

In a similar manner, using Lemma 2, we can prove

Lemma 4: If Δ is square free, the integer solutions of $\Delta(x^2 + 4) = z^2$ are precisely the pairs $(x, z) = (\pm V_{2n+1}, \pm \Delta U_{2n+1})$.

Results for the sequences (1.4) which parallel (2.1)-(2.6) are as follows:

$$v_m^2 - 4 = Du_m^2, \tag{2.7}$$

$$u_n v_m + v_n u_m = 2u_{n+m}, \tag{2.8}$$

$$u_n v_m - v_n u_m = 2u_{n-m}, \tag{2.9}$$

$$v_n v_m + Du_m u_n = 2v_{n+m}, \tag{2.10}$$

$$v_n v_m - Du_n u_m = 2v_{n-m}. \tag{2.11}$$

For completion, we state the following lemma, which is the second part of McDaniel's Corollary 1.

Lemma 5: The integer solutions of $Dx^2+4=z^2$ are precisely the pairs $(x,z)=(\pm u_n, \pm v_n)$.

Now, using Lemma 5, and following the method of proof of Lemma 3, it is easy to prove

Lemma 6: If D is square free, the integer solutions of $D(x^2-4)=z^2$ are precisely the pairs $(x,z)=(\pm v_n, \pm Du_n)$.

3. CONICS CHARACTERIZING THE SEQUENCES (1.3)

We now give a sequence of theorems concerning pairs of conics whose integer points are derived from the sequences (1.3). In the proofs we must recall that

$$\sqrt{a^2}=\begin{cases} a, & a \geq 0, \\ -a, & a<0. \end{cases}$$

Theorem 1: If m is a fixed even integer, then the points with integer coordinates on the conics $y^2-V_mxy+x^2 \pm U_m^2=0$ are precisely the pairs $(x,y)=\pm(U_n, U_{n+m})$.

Proof: Consider first the conic $y^2-V_mxy+x^2+U_m^2=0$. Regarding this as a quadratic equation in y, and making use of (2.1), we obtain

$$y=\frac{V_m x \pm U_m\sqrt{\Delta x^2-4}}{2}.$$

From Lemma 2, integer points can arise only when $x=\pm U_{2n+1}$. Now, using (2.3) and (2.4), we see that the integer points are $(x,y)=\pm(U_{2n+1}, U_{2n+1+m})$ together with the points $(x,y)=\pm(U_{2n+1}, U_{2n+1-m})$, where n ranges over all integers. Since these sets coincide, we consider only the first.

Proceeding in the same manner, and making use of (2.1), Lemma 1, (2.3), and (2.4), we see that the integer points on the conic $y^2-V_mxy+x^2-U_m^2=0$ are $(x,y)=\pm(U_{2n}, U_{2n+m})$. This completes the proof. □

We now state three additional theorems, each of which can be proved using the results of Section 2. Since the proofs are similar to the proof of Theorem 1, we refrain from giving them here.

Theorem 2: If m is a fixed odd integer, then the points with integer coordinates on the conics $y^2-V_mxy-x^2 \pm U_m^2=0$ are precisely the pairs $(x,y)=\pm(U_n, U_{n+m})$.

Theorem 3: If m is a fixed even integer and Δ is square free, then the points with integer coordinates on the conics $y^2-V_mxy+x^2 \pm \Delta U_m^2=0$ are precisely the pairs $(x,y)=\pm(V_n, V_{n+m})$.

Theorem 4: If m is a fixed odd integer and Δ is square free, then the points with integer coordinates on the conics $y^2-V_mxy-x^2 \pm \Delta U_m^2=0$ are precisely the pairs $(x,y)=\pm(V_n, V_{n+m})$.

We remark that Theorem 2 generalizes McDaniel's Theorem 1, and Theorem 4 generalizes McDaniel's Corollary 2.

4. CONICS CHARACTERIZING THE SEQUENCES (1.4)

Next, we state two theorems concerning conics whose integer points are derived from the sequences (1.4). Each can be proved by following the method of proof of Theorem 1, while making use of the appropriate results from Section 2.

Theorem 5: If m is any fixed integer, then the points with integer coordinates on the conic $y^2 - v_m xy + x^2 - u_m^2 = 0$ are precisely the pairs $(x, y) = \pm(u_n, u_{n+m})$.

Theorem 6: If m is any fixed integer and D is square free, then the points with integer coordinates on the conic $y^2 - v_m xy + x^2 + Du_m^2 = 0$ are precisely the pairs $(x, y) = \pm(v_n, v_{n+m})$.

We note that Theorem 5 generalizes McDaniel's Theorem 2, and Theorem 6 generalizes McDaniel's Corollary 3.

5. AN INTERESTING EXAMPLE

If Δ is not square free, it is easy to show by substitution, using Binet forms, that the stated solutions in Theorems 3 and 4 remain as solutions. The same is true of Theorem 6. However, as McDaniel observes, other solutions may occur. He cites the example

$$y^2 - 4xy - x^2 \pm 20 = 0. \tag{5.1}$$

The conics (5.1) provide an example of the conics in Theorem 4 where $P = 4$, $m = 1$, and $\Delta = 20 = 2^2.5$ is not square free. Now $(x, y) = (1, 7)$ is a solution of (5.1), but $V_n \neq 1$ for any n. Observe, however, that the conics (5.1) may be written as

$$y^2 - L_3 xy - x^2 \pm 5F_3^2 = 0. \tag{5.2}$$

This is an instance of Theorem 4 in which $P = 1$, $m = 3$, and $\Delta = 5$ is square free. Hence, the solutions are precisely $(x, y) = \pm(L_n, L_{n+3})$.

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