STRONGLY MAGIC SQUARES

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I. INTRODUCTION

Consider the classical 4×4 magic square

$$M_{\text{Classical}} = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}.$$

Among the 2×2 subsquares that can be formed within $M_{\text{Classical}}$, there are only five with the property that their entries add to the magic constant 34. These are the four corner squares and the central one:

$$\begin{bmatrix} 16 & 2 \\ 5 & 11 \end{bmatrix}, \begin{bmatrix} 3 & 13 \\ 10 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 7 \\ 4 & 14 \end{bmatrix}, \begin{bmatrix} 6 & 12 \\ 15 & 1 \end{bmatrix}, \begin{bmatrix} 11 & 10 \\ 7 & 6 \end{bmatrix}.$$

If wrap-arounds are allowed, one more such subsquare arises, namely,

$$\begin{bmatrix} 1 & 4 \\ 13 & 16 \end{bmatrix},$$

built from the four corners.

Compare this to the square

$$M^* = \begin{bmatrix} 9 & 16 & 5 & 4 \\ 7 & 2 & 11 & 14 \\ 12 & 13 & 8 & 1 \\ 6 & 3 & 10 & 15 \end{bmatrix}$$

which has the stronger property that all sixteen 2×2 subsquares (allowing wrap-arounds) have entries adding to 34. What other magic squares have this stronger property?

Suppose M is a 4×4 magic square. That is, the sixteen entries are a permutation of the set $[1, 2, 3, \dots, 16]$ and all the row-sums and column-sums equal 34. Writing this as

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

we define M to be a *strongly magic square* if, in addition, $a_{m,n} + a_{m,n+1} + a_{m+1,n} + a_{m+1,n+1} = 34$ whenever $1 \le m, n \le 3$. We will derive several further properties of strongly magic squares. For example, it follows that all the wrap-around 2×2 subsquares, like

$$\begin{bmatrix} a_{14} & a_{11} \\ a_{24} & a_{21} \end{bmatrix},$$

also have entries adding to 34. Moreover, we will classify all the strongly magic squares, showing that there are exactly 384 of them. We also define a group of transformations by which each strongly magic square can be transformed into all the other 383 strongly magic squares.

II. SPECIAL PROPERTIES OF STRONGLY MAGIC SQUARES

1. In a strongly magic square

$$M_s = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

$$a_{11} + a_{12} = a_{23} + a_{24} = a_{31} + a_{32} = a_{43} + a_{44} = A \tag{1}$$

and

$$a_{13} + a_{14} = a_{21} + a_{22} = a_{33} + a_{34} = a_{41} + a_{42} = 34 - A. \tag{2}$$

This property follows directly from the definition of strongly magic squares.

2. The 3×3 Square Property.

Consider any 3×3 square formed within a strongly magic square M_s. The sum of the four corners of this square is 34 and the sum of each diagonally opposite corner pair is 17.

Let $C_1, C_2, C_3,$ and C_4 be the corner elements of any 3×3 subsquare of M_s. Then

$$C_1 + C_4 = C_2 + C_3 = 17.$$

Proof: Each of the corners $C_1, C_2, C_3,$ and C_4 of any 3×3 subsquare can be considered as a corner of a 2×2 subsquare, three of them being corner squares and one being the inner central square. For example, consider the 3×3 square

$$M_3' = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}:$$

$$C_1 = a_{22}, \text{ a corner of} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \text{(inner central square);}$$

$$C_2 = a_{24}, \text{ a corner of} \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \text{(corner square);}$$

$$C_3 = a_{42}, \text{ a corner of} \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \text{(corner square);}$$

$$C_4 = a_{44}, \text{ a corner of} \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \text{(corner square).}$$

The corners of any 3×3 square of M_s can therefore be written as

$$\begin{aligned} C_1 &= 34 - S_1, \\ C_2 &= 34 - S_2, \\ C_3 &= 34 - S_3, \\ C_4 &= 34 - S_4, \end{aligned}$$

where S_1, S_2, S_3, and S_4 are the sums of the other three elements of the respective 2×2 magic squares. For the particular 3×3 square M_3',

$$\begin{aligned} S_1 &= a_{23} + a_{32} + a_{33}, \\ S_2 &= a_{13} + a_{14} + a_{23}, \\ S_3 &= a_{31} + a_{32} + a_{41}, \\ S_4 &= a_{33} + a_{34} + a_{43}. \end{aligned}$$

The sum of the corners can be written as

$$C_1 + C_2 + C_3 + C_4 = 4 \times 34 - (S_1 + S_2 + S_3 + S_4).$$

Regrouping the terms that constitute S_1, S_2, S_3, and S_4 in the sum $S_1 + S_2 + S_3 + S_4$, it can be shown that $S_1 + S_2 + S_3 + S_4$ is the sum of a row, a column, and a diagonal of the 4×4 square, each of which is equal to 34. For example, for M_3',

$$\begin{aligned} S_1 + S_2 + S_3 + S_4 &= (a_{31} + a_{32} + a_{33} + a_{34}) + (a_{13} + a_{23} + a_{33} + a_{43}) + (a_{14} + a_{23} + a_{32} + a_{41}) \\ &= 34 + 34 + 34. \end{aligned}$$

Therefore,

$$C_1 + C_2 + C_3 + C_4 = 4 \times 34 - 3 \times 34 = 34.$$

This is true for the classical magic square also.

In the case of strongly magic squares, two of the 2×2 squares that contain C_1, C_2, C_3, and C_4 can be chosen to be those formed by the inner two rows and columns (which have the magic property in M_s and not in $M_{\text{Classical}}$). For example, in the 3×3 square M_3', the corners C_2 and C_3 can be considered a part of the 2×2 squares

$$\begin{bmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{bmatrix} \text{ and } \begin{bmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{bmatrix},$$

respectively, in this case,

$$\begin{aligned} S_1 &= a_{23} + a_{32} + a_{33}, \\ S_2 &= a_{23} + a_{33} + a_{34}, \\ S_3 &= a_{32} + a_{33} + a_{43}, \\ S_4 &= a_{33} + a_{34} + a_{43}; \end{aligned}$$

$$S_2 + S_3 = S_1 + S_4,$$

i.e.,

$$C_1 + C_4 = C_2 + C_3.$$

Since $C_1 + C_2 + C_3 + C_4 = 34$, we have $C_1 + C_4 = C_2 + C_3 = 17$.

3. Triangular Property

Form any triangle each side of which is made of three numbers of the 4×4 strongly magic square. Examples of such triangles are shown below:

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{matrix} \qquad\qquad \begin{matrix} & & a_{24} \\ & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{matrix}$$

In M_s the sum of the six numbers along the sides of the triangle is the same for all such triangles and equal to 51. This also can be shown to follow from the additional 2×2 magic property.

Proof: Of the six numbers that constitute the sum, three can be considered as a part of a row or column of the M_s, and the other three as part of a 2×2 magic square. Let S_1 be the sum of the three numbers that are part of the row or column and S_2 that of three numbers that are part of the 2×2 square.

Since the sum of the numbers of the row or column as well as the sum of the numbers of the 2×2 square are equal to 34,

$$S_1 = 34 - N_1,$$
$$S_2 = 34 - N_2,$$

where N_1 is the remaining element of the row/column and N_2 is the remaining element of the 2×2 magic square. It is easy to see that N_1 and N_2 always form the opposite corners of a 3×3 square whose sum is 17. Therefore, the sum of the sides of the triangle can be written as

$$S = S_1 + S_2 = 68 - (N_1 + N_2) = 68 - 17 = 51.$$

III. TRANSFORMATIONS THAT PRESERVE THE STRONGLY MAGIC PROPERTY

There exist several transformations which, when applied to a strongly magic square yields another strongly magic square.

Some of these transformations along with the notations we use to represent them later in the paper are given below.

1) Cycling of rows (cyc R) or columns (cyc C).

2) Interchange of columns 1 and 3 ($C_{1\leftrightarrow 3}$) or rows 1 and 3 ($R_{1\leftrightarrow 3}$).

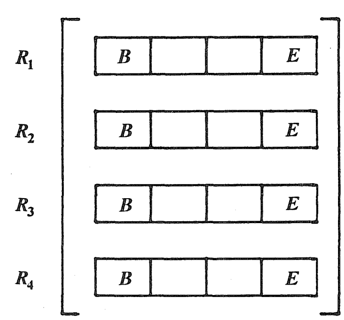
3) Interchange of columns 2 and 4 ($C_{2\leftrightarrow 4}$) or rows 2 and 4 ($R_{2\leftrightarrow 4}$).

4) Diagonal reflections (DRA on the ascending diagonal and DRD on the descending diagonal).

5) Replacement of every element x by $17 - x$.

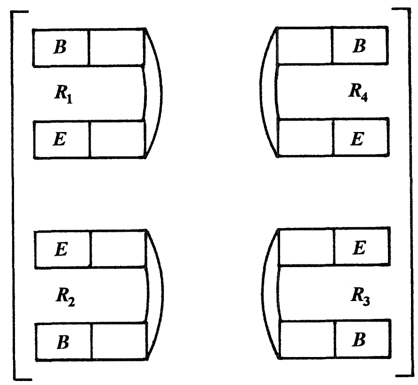
6) "Twisting" of rows (TWR) or columns (TWC) which is defined below:

The row twist of a square M is obtained by curling the first row of M into the upper left corner of TWR(M), the second row of M into the lower left corner of TWR(M), etc. Consider a strongly magic square with rows R_1, R_2, R_3, and R_4:



B and E are the beginning and end of the rows.

The "twisting" transformation is defined to yield the following square:



For example, the row-twisting transformation on M^* yields

$$\mathrm{TWR}(M^*) = \begin{bmatrix} 9 & 16 & 3 & 6 \\ 4 & 5 & 10 & 15 \\ 14 & 11 & 8 & 1 \\ 7 & 2 & 13 & 12 \end{bmatrix};$$

column-twisting M^* yields

$$\mathrm{TWC}(M^*) = \begin{bmatrix} 9 & 7 & 14 & 4 \\ 6 & 12 & 1 & 15 \\ 3 & 13 & 8 & 10 \\ 16 & 2 & 11 & 5 \end{bmatrix}.$$

It can be noted that $\mathrm{TWC}(M) = \mathrm{TWR}(M^{\mathrm{T}})$, where M^{T} is the transpose of the matrix M.

IV. THE TOTAL NUMBER OF DISTINCT STRONGLY MAGIC SQUARES

Only 384 distinct strongly magic squares can be formed from the set of numbers [1, 2, ..., 16].

Proof: We saw in Section II that

$$a_{11} + a_{12} = a_{23} + a_{24} = a_{31} + a_{32} = a_{43} + a_{44} = A \tag{1}$$

and

$$a_{13} + a_{14} = a_{21} + a_{22} = a_{33} + a_{34} = a_{41} + a_{42} = 34 - A. \tag{2}$$

Going through all possible values of A, it can be seen that A can take only eight values, namely, 9, 13, 15, 16, 18, 19, 21, and 25. Any other value of A would lead to at least two of the elements of the magic square being equal.

We now show that each value of A leads to 48 and only 48 strongly magic squares.

Consider the elements a_{12}, a_{24}, a_{31}, and a_{43} of a strongly magic square M_s. The strongly magic property implies that

$$\left.\begin{array}{l} a_{13} = 17 - a_{31} \\ a_{21} = 17 - a_{43} \\ a_{34} = 17 - a_{12} \\ a_{42} = 17 - a_{24} \end{array}\right\} \text{ from the } 3\times 3 \text{ square property,}$$

$$\left.\begin{array}{l} a_{11} = A - a_{12} \\ a_{23} = A - a_{24} \\ a_{32} = A - a_{31} \\ a_{44} = A - a_{43} \end{array}\right\},$$

$$\left.\begin{array}{l} a_{14} = 34 - A - a_{13} = 34 - A - (17 - a_{31}) = a_{31} - (A - 17) \\ a_{22} = 34 - A - a_{21} = 34 - A - (17 - a_{43}) = a_{43} - (A - 17) \\ a_{33} = 34 - A - a_{34} = 34 - A - (17 - a_{12}) = a_{12} - (A - 17) \\ a_{41} = 34 - A - a_{42} = 34 - A - (17 - a_{24}) = a_{24} - (A - 17) \end{array}\right\} \text{ from equations (1) and (2).}$$

Thus, for a given value of A, M_s can be written as

$$M_s(A) = \begin{bmatrix} (A - a_{12}) & a_{12} & a'_{31} & (A - a_{31})' \\ a'_{43} & (A - a_{43})' & (A - a_{24}) & a_{24} \\ a_{31} & (A - a_{31}) & (A - a_{12})' & a'_{12} \\ (A - a_{24})' & a'_{24} & a_{43} & (A - a_{43}) \end{bmatrix},$$

where the notation x' means $17 - x$. Additionally, in order that all the elements are distinct and positive, further conditions have to be satisfied by the set $(a_{12}, a_{24}, a_{31}, a_{43})$. These conditions depend on the value of A. For example, when $A = 25$, a_{12}, a_{24}, a_{31}, and a_{43} can take on values between 9 and 16 only. Also, $a_{12} + a_{43} = a_{24} + a_{31} \neq 25$ because, if the sum is equal to 25, the elements of M_s (25) cannot be distinct.

By considering all possible number pair sets (a_{12}, a_{43}), (a_{24}, a_{31}) satisfying the above two conditions, for $A = 25$, it is seen that they are:

$$\begin{array}{l} (16, 11), (15, 12);\ \ (16, 10), (14, 12);\ \ (15, 9), (13, 11); \\ (14, 9), (13, 10);\ \ (16, 13), (15, 14);\ \ (9, 12), (10, 11); \end{array}$$

and all the permutations possible within each set. From each of the above six sets, eight permutations are possible, leading to $8 \times 6 = 48$ possibilities for the set $[a_{12}, a_{24}, a_{31}, a_{43}]$.

Thus, we have proved that with $A = 25$, 48, and only 48, strongly magic squares can be obtained. Similarly, it is possible to obtain exactly 48 strongly magic squares from each of the other seven values of A. However, since this is a tedious procedure to prove directly, we follow a different approach. We show in the following that performing certain sequences of transformations on each of the 48 strongly magic squares for any value of A, we can get all strongly magic squares with the other seven values of A.

These transformations can be shown to be sequences of 3 basic transformations, namely,

$$\begin{array}{l} T_1 = C_{1\leftrightarrow 3}, \\ T_2 = \text{ DRA}, \\ T_3 = C_{1\leftrightarrow 3} + C_{2\leftrightarrow 4}. \end{array}$$

Applying T_1, T_2, and T_3 on any strongly magic square, we can form seven other strongly magic squares each with a distinct value of A. Let

$$M_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

$$M_2 = T_1(M_1),$$
$$M_3 = T_2(M_1),$$
$$M_4 = T_1(M_3),$$
$$M_{5\text{ to }8} = T_3(M_{1\text{ to }4}).$$

The values of A for the squares $M_{1\text{ to }8}$ are given below:

$$\begin{aligned} A_1 &= a_{11} + a_{12}, \\ A_2 &= a'_{31} + a_{12}, \\ A_3 &= a'_{43} + (A_1 - a_{12}), \\ A_4 &= a_{31} + a'_{43}, \\ A_5 &= a'_{31} + (A_1 - a_{31})', \\ A_6 &= (A_1 - a_{12}) + (A_1 - a_{31})', \\ A_7 &= a_{31} + (A_1 - a_{24})', \\ A_8 &= (A_1 - a_{12}) + (A_1 - a_{24})', \end{aligned}$$

where $x' = 17 - x$.

Remembering that A can have only eight possibilities and that with the $A = 25$ we can have only 48 strongly magic squares; we can see that we get $48 \times 8 = 384$ strongly magic squares. Now, the above transformations applied to any strongly magic square with any other value of A will also yield strongly magic squares having the seven other possibilities for A which includes $A = 25$. If there are N possible magic squares with a certain value of $A \neq 25$, we can get $8N$ strongly magic squares by performing the transformations on each of the N squares. Thus, there will be N strongly magic squares for each value of A, including $A = 25$. We have already proved that there can be only 48 strongly magic squares with $A = 25$.

Thus, $N = 48$ for any value of A, and we have proved that there are exactly 384 strongly magic squares formed.

Equivalence of Strongly Magic Squares

Two strongly magic squares are defined as equivalent if one can be transformed to the other by a transformation or a sequence of transformations. It is shown below that each strongly magic square can be transformed into all the other 383 squares.

Take any strongly magic square,

$$M_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Then 24 distinct strongly magic squares with a_{11} as the first element can be formed from M_1 be applying some of the transformations mentioned in Section III in sequence:

$$
\begin{aligned}
&M_2 = \mathrm{TWR}(M_1);\\
&M_3 = \mathrm{TWR}(M_2);\\
&M_{4\text{ to }6} = \mathrm{DRA}(M_{1\text{ to }3});\\
&M_{7\text{ to }9} = C_{2\leftrightarrow 4}(M_{1\text{ to }3});\\
&M_{10\text{ to }12} = C_{2\leftrightarrow 4}(M_{4\text{ to }6});\\
&M_{13\text{ to }24} = R_{2\leftrightarrow 4}(M_{1\text{ to }12}).
\end{aligned}
$$

Note that a_{11} can be any of the 16 numbers from 1 to 16 because any of these numbers can be brought to the (1, 1) position by an appropriate sequence of row and column cycling. Each of these can then be transformed to 24 distinct strongly magic squares by the above mentioned transformations. Thus, one can obtain all the $384 = 16 \times 24$ strongly magic squares from any strongly magic square, i.e., all strongly magic squares are equivalent to all other strongly magic squares.

It is also clear from the above that there are 24 distinct strongly magic squares for any one position of a number in the square. It has already been shown that there are 24 strongly magic squares with any number occupying the (1, 1) position. Performing appropriate sequences of row and column cycling on these 24 squares, this number can be brought to the desired position, i.e., 24 strongly magic squares can be formed for a particular position of any number.

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