ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-532 *Proposed by Paul S. Bruckman, Highwood, IL*

Let $V_n = V_n(x)$ denote the generalized Lucas polynomials defined as follows: $V_0 = 2$; $V_1 = x$; $V_{n+2} = xV_{n+1} + V_n$, $n = 0, 1, 2, \ldots$. If n is an odd positive integer and y is any real number, find all (exact) solutions of the equation: $V_n(x) = y$.

H-533 *Proposed by Andrej Dujella, University of Zagreb, Croatia*

Let $Z(n)$ be the entry point for positive integers n. Prove that $Z(n) \le 2n$ for any positive integer n. Find all positive integers n such that $Z(n) = 2n$.

H-534 *Proposed by Piero Filipponi, Rome, Italy*

An interesting question posed to me by Evelyn Hart (Colgate University, Hamilton, NY) led me to pose, in turn, the following two problems to the readers of *The Fibonacci Quarterly*.

Problem A: For k a fixed positive integer, let n_k be *any* integer representable as

$$n_k = \sum_{j=1}^{k} v_j F_j, \tag{1}$$

where v_j equals either j or zero.

Remarks:

(i) Clearly, we have that $0 \le n_k \le f(k) = (k+1)F_{k+2} - F_{k+4} + 2$ (see Hoggatt's identity I_{40}).

(ii) In general, the representation (1) is not unique, as shown by the following example: $91 = 7F_7 = 6F_6 + 5F_5 + 4F_4 + 3F_3$.

(iii) Not all integers can be represented as (1), 4, 5, 10, 11, 16, 17, 22, 23, and 24 being the smallest among such integers.

Let $S(k)$ be the number of all n_k. Is it possible to evaluate $\lim_{k\to\infty} \frac{S(k)}{f(k)}$?

Problem B: Is it possible to characterize the set of all positive integers k for which kF_k is representable as

$$kF_k = \sum_{j=1}^{k-1} v_j F_j,$$

where v_j is as in Problem A?

Remarks:

(i) Since $kF_k > \sum_{j=1}^{k-1} jF_j$ for $k \le 6$, we must have $k \ge 7$. In fact, $7F_7 = 91$ can be represented in this form [see Remark (ii) in Problem A].

(ii) The numerical inspection of earliest cases shows that other values of k are 10, 11, 12, 13, 15, and 16. As an example, we have: $16F_{16} = 15F_{15} + 14F_{14} + 11F_{11} + 9F_9 + 6F_6 + 5F_5 + 3F_3$.

H-535 ***Proposed by Piero Filipponi & Adina Di Porto, Rome, Italy***

For given positive integers n and m, find a closed form expression for $\sum_{k=1}^{n} k^m F_k$.

Conjecture by the proposers:

$$\Sigma_{m,n} = \sum_{k=1}^{n} k^m F_k = p_1^{(m)}(n)F_{n+1} + p_2^{(m)}(n)F_n + C_m, \tag{1}$$

where $p_1^{(m)}(n)$ and $p_2^{(m)}(n)$ are polynomials in n of degree m,

$$p_1^{(m)}(n) = \sum_{i=0}^{m} (-1)^i a_{m-i}^{(m)} n^{m-i}, \quad p_2^{(m)}(n) = \sum_{i=0}^{m} (-1)^i b_{m-i}^{(m)} n^{m-i}, \tag{2}$$

the coefficients $a_k^{(m)}$ and $b_k^{(m)}$ $(k = 0, 1, \ldots, m)$ are positive integers, and C_m is an integer.

On the basis of the well-known identity

$$\Sigma_{1,n} = (n-2)F_{n+1} + (n-1)F_n + 2, \tag{3}$$

which is an alternate form of Hoggatt's identity I_{40}, the above quantities can be found recursively by means of the following algorithm:

1. $p_1^{(m+1)}(n) = (m+1)\int p_1^{(m)}(n)\,dn + (-1)^{m+1} a_0^{(m+1)}, \; p_2^{(m+1)}(n) = (m+1)\int p_2^{(m)}(n)\,dn + (-1)^{m+1} b_0^{(m+1)}$.

2. $a_0^{(m+1)} = \sum_{i=1}^{m+1} (a_i^{(m+1)} + b_i^{(m+1)})$.

3. $b_0^{(m+1)} = \sum_{i=1}^{m+1} a_i^{(m+1)}$.

4. $C_{m+1} = (-1)^m a_0^{(m+1)}$.

Example: The following results were obtained using the above algorithm:

$$\Sigma_{2,n} = (n^2 - 4n + 8)F_{n+1} + (n^2 - 2n + 5)F_n - 8;$$

$$\Sigma_{3,n} = (n^3 - 6n^2 + 24n - 50)F_{n+1} + (n^3 - 3n^2 + 15n - 31)F_n + 50;$$

$$\Sigma_{4,n} = (n^4 - 8n^3 + 48n^2 - 200n + 416)F_{n+1} + (n^4 - 4n^3 + 30n^2 - 124n + 257)F_n - 416;$$

$$\begin{aligned}\Sigma_{5,n} = {} & (n^5 - 10n^4 + 80n^3 - 500n^2 + 2080n - 4322)F_{n+1} \\ & + (n^5 - 5n^4 + 50n^3 - 310n^2 + 1285n - 2671)F_n + 4322.\end{aligned}$$

Remarks:

(i) These results can obviously be proved by induction on n.

(ii) It can be noted that, using the same algorithm, $\Sigma_{1,n}$ can be obtained by the identity $\Sigma_{0,n} = F_{n+1} + F_n - 1$.

(iii) It appears that $a_k^{(m+k)} / b_k^{(m+k)} = \text{const.} = a_0^{(m)} / b_0^{(m)}$, $(k = 1, 2, \ldots)$ and $\lim_{m\to\infty} a_0^{(m)} / b_0^{(m)} = \alpha$.

SOLUTIONS

Limits

H-514 *Proposed by Juan Pla, Paris, France*
(Vol. 34, no. 4, August 1996)

I) Let (L_n) be the generalized Lucas sequence of the recursion $U_{n+2}-2aU_{n+1}+U_n=0$ with a real such that $a>1$. Prove that

$$\lim_{n\to+\infty}\frac{L_2L_{2^2}L_{2^3}\dots L_{2^n}}{L_{2^{n+1}}}=\frac{1}{4}\frac{1}{a\sqrt{a^2-1}}.$$

II) Show that the above expression has a limit when (L_n) is the *classical* Lucas sequence.

Solution by H.-J. Seiffert, Berlin, Germany

Let (L_n) be the generalized Lucas sequence of the recursion $U_{n+2}-2aU_{n+1}+bU_n=0$ with a and b real such that $a>0$ and $a^2>b$. Then L_n has the Binet form $L_n=\alpha^n+\beta^n$, $n\in N_0$, where $\alpha=a+\sqrt{a^2-b}$ and $\beta=a-\sqrt{a^2-b}$. Let $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)$, $n\in N_0$. Since $\alpha>|\beta|$ by $a>0$ and $a^2>b$, we have

$$\lim_{n\to+\infty}\frac{F_n}{L_n}=\lim_{n\to+\infty}\frac{\alpha^n-\beta^n}{(\alpha-\beta)(\alpha^n+\beta^n)}=\frac{1}{\alpha-\beta}\lim_{n\to+\infty}\frac{1-(\beta/\alpha)^n}{1+(\beta/\alpha)^n}=\frac{1}{\alpha-\beta}$$

or

$$\lim_{n\to+\infty}\frac{F_n}{L_n}=\frac{1}{2\sqrt{a^2-b}}. \tag{1}$$

It is easily verified that $F_{2n}-F_nL_n$, $n\in N_0$. Now, a simple induction argument yields

$$L_{2k}L_{2^2k}L_{2^3k}\dots L_{2^nk}=\frac{F_{2^{n+1}k}}{F_{2k}},\quad k\in N,\ n\in N_0.$$

Hence, by (1),

$$\lim_{n\to+\infty}\frac{L_{2k}L_{2^2k}L_{2^3k}\dots L_{2^nk}}{L_{2^{n+1}k}}=\frac{1}{2F_{2k}\sqrt{a^2-b}} \tag{2}$$

for all $k\in N$. In the special case $k=1$, this limit is $1/(4a\sqrt{a^2-b})$. The more special case $b=1$ and $(a>1)$ solves the first part of the proposal. Taking $a=1/2$, $b=-1$, and $k=1$, (2) gives the value $1/\sqrt{5}$ for the limit considered in the second part of the proposal.

Also solved by P. Bruckman, C. Georgiou, J. Koštál, and the proposer.

Some Entry

H-515 *Proposed by Paul S. Bruckman, Highwood, IL*
(Vol. 34, no. 4, August 1996)

For all primes $p\neq 2,5$, let $Z(p)$ denote the entry-point of p in the Fibonacci sequence. It is known that $Z(p)\,|\,(p-(\frac{5}{p}))$. Let $a(p)=(p-(\frac{5}{p}))/Z(p)$, $q=\frac{1}{2}(p-(\frac{5}{p}))$. Prove that if $p\equiv 1$ or 9 (mod 20) then

$$F_{q+1}\equiv(-1)^{\frac{1}{2}(q+a(p))}\pmod p. \tag{*}$$

Solution by H.-J. Seiffert, Berlin, Germany

We will use the easily verifiable equations

$$F_{2n+1} = F_n L_{n+1} + (-1)^n \quad \text{and} \quad F_{2n+1} = F_{n+1} L_n - (-1)^n, \tag{1}$$

where n is any integer, and the following known results:

$$\left(\tfrac{5}{p}\right) = 1 \text{ if } p \equiv 1 \text{ or } 9 \pmod{10}, \tag{2}$$

$$p \mid F_q \text{ if and only if } p \equiv 1 \pmod 4, \tag{3}$$

$$Z(p) \mid m \text{ if and only if } p \mid F_m, \tag{4}$$

where $p \neq 5$ denotes an odd prime and m a positive integer.

Let p be an odd prime such that $p \equiv 1$ or $9 \pmod{20}$. From (2), we have $\left(\frac{5}{p}\right) = 1$, so that $q = \frac{1}{2}(p-1)$.

First, suppose that $p \mid F_{q/2}$. Then $Z(p) \mid q/2$ by (4), which yields $a(p) \equiv 0 \pmod 4$. Using the left equation of (1) with $n = q/2$, it follows that

$$F_{q+1} = F_{q/2} L_{\frac{1}{2}(q+2)} + (-1)^{q/2} \equiv (-1)^{q/2} \pmod p,$$

which proves $(*)$ in this case.

If $p \nmid F_{q/2}$, then $p \mid L_{q/2}$, since p divides $F_q = F_{q/2} L_{q/2}$, by (3). Since $Z(p) \mid q$ and $Z(p) \nmid q/2$, by (4), we have $a(p) \equiv 2 \pmod 4$. Using the right equation of (1) with $n = q/2$, we obtain

$$F_{q+1} = F_{\frac{1}{2}(q+2)} L_{q/2} - (-1)^{q/2} \equiv (-1)^{\frac{1}{2}(q+2)} \pmod p,$$

proving $(*)$ in such case.

Also solved by the proposer.

Mod Squad

H-516 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 34, no. 4, August 1996)

Given p an odd prime, let $\bar{k}(p)$ denote the *Lucas period (mod p)*, that is, $\bar{k}(p)$ is the smallest positive integer $m = m(p)$ such that $L_{m+n} \equiv L_n \pmod p$ for all integers n. Prove the following:

(a) Let $u = u(p)$ denote the smallest positive integer such that $\alpha^u \equiv \beta^u \equiv 1 \pmod p$. Then $u = m = \bar{k}(p)$.

(b) $\bar{k}(p)$ is even for all (odd) p.

(c) $p \equiv 1 \pmod{\bar{k}(p)}$ iff $p = 5$ or $p \equiv \pm 1 \pmod{10}$.

(d) $p \equiv -1 + \frac{1}{2}\bar{k}(p) \pmod{\bar{k}(p)}$ iff $p = 5$ or $p \equiv \pm 3 \pmod{10}$.

Solution by the proposer

We will use the following fairly well-known result that $\alpha^p \equiv \alpha$, $\beta^p = \beta \pmod p$ iff $p = 5$ or $p \equiv \pm 1 \pmod{10}$, while $\alpha^p \equiv \beta$, $\beta^p \equiv \alpha \pmod p$ iff $p = 5$ or $p \equiv \pm 3 \pmod{10}$. Also, we shall use the easily demonstrable result that $\bar{k}(p) = 4$ iff $p = 5$. The first result implies that u always exists.

Proof of (a): If $p=5$, then $\alpha \equiv \beta \equiv 2^{-1} \equiv -2 \pmod 5$; we see readily that $u=m=\bar{k}(5)=4$.

If $p \neq 5$, suppose the congruence in the statement of the problem. Then, for all integers n, we have $\alpha^{u+n} \equiv \alpha^n$, $\beta^{u+n} \equiv \beta^n \pmod p$, which implies (by addition) $L_{u+n} \equiv L_n \pmod p$. This, in turn, implies that $m|u$. On the other hand, $L_{m+n} \equiv L_n \pmod p$ for all integers n, and in particular for $n=-1$, 0, and 1; hence, $L_{m-1} \equiv L_{-1} \equiv -1$, $L_m \equiv L_0 \equiv 2$, $L_{m+1} \equiv L_1 \equiv 1 \pmod p$. Then $L_{m-1} + L_{m+1} = 5F_m = 5^{1/2}(\alpha^m - \beta^m) \equiv 0 \pmod p$, so $\alpha^m \equiv \beta^m \pmod p$. Since $L_m = \alpha^m + \beta^m \equiv 2 \pmod p$, we have $\alpha^m \equiv \beta^m \equiv 1 \pmod p$. From this, it follows that $u|m$. Hence, $u=m$. Q.E.D.

Proof of (b): Since $\alpha^m \equiv \beta^m \equiv 1 \pmod p$, we have that $(\alpha\beta)^m = (-1)^m \equiv 1 \pmod p$, which implies that $m = \bar{k}(p)$ must be even.

Proof of (c): Since $\bar{k}(p)=4$ iff $p=5$, we see that the first congruence in the statement of (c) is satisfied by $p=5$. Suppose $p \neq 5$ and $p \equiv 1 \pmod{\bar{k}(p)}$. Then $\alpha^p \equiv \alpha$, $\beta^p \equiv \beta \pmod p$, which implies $p \equiv \pm 1 \pmod{10}$.

Conversely, if $p \equiv \pm 1 \pmod{10}$, then $\alpha^p \equiv \alpha$, $\beta^p \equiv \beta \pmod p$. so $\alpha^{p-1} \equiv \beta^{p-1} \equiv 1 \pmod p$. Then $\bar{k}(p)|(p-1)$ or $p \equiv 1 \pmod{\bar{k}(p)}$.

Proof of (d): We see that the first congruence in the statement of (d) is satisfied by $p=5$. Suppose that it is satisfied by $p \neq 5$. Then $\bar{k}(p)|(2p+2)$, $\bar{k}(p)\nmid(p+1)$, so $\alpha^{p+1} \equiv \beta^{p+1} \equiv -1 \pmod p$; for if $\alpha^{p+1} \equiv -\beta^{p+1} \equiv \pm 1 \pmod p$, then $(\alpha\beta)^{p+1} \equiv -1$, which is absurd, since $(-1)^{p+1}=1$ (for odd p). Then $\alpha^p \equiv \beta$, $\beta^p \equiv \alpha \pmod p$, which implies $p \equiv \pm 3 \pmod{10}$.

Conversely, if $p \equiv \pm 3 \pmod{10}$, then $\alpha^p \equiv \beta$, $\beta^p \equiv \alpha \pmod p$, which implies $\alpha^{p+1} \equiv \beta^{p+1} \equiv -1$, $\alpha^{2p+2} \equiv \beta^{2p+2} \equiv 1 \pmod p$. Therefore, $\bar{k}(p)|(2p+2)$, $\bar{k}(p)\nmid(p+1)$, which implies $p \equiv -1 + \frac{1}{2}\bar{k}(p) \pmod{\bar{k}(p)}$.

Also solved by L. A. G. Dresel.

Divide and Conquer

H-517 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 34, no. 5, November 1996)***

Given a positive integer n, define the sums $P(n)$ and $Q(n)$ as follows:

$$P(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) L_d, \quad Q(n) = \sum_{d|n} \Phi\left(\frac{n}{d}\right) L_d, \tag{1}$$

where μ and Φ are the Möbius and Euler functions, respectively. Show that $n|P(n)$ and $n|Q(n)$.

Solution by H.-J. Seiffert, Berlin, Germany

It is well known that

$$L_{kp^r} \equiv L_{kp^{r-1}} \pmod{p^r} \text{ if } p \text{ is a prime and } k, r \in N. \tag{1}$$

Let $n \in N$ be divisible by the prime p. Then there exist $m, e \in N$ such that $p \nmid m$ and $n = mp^e$.

Using $\mu(d)=0$ if $d \in N$ and $p^2|d$, $\mu(jp) = -\mu(j)$ if $j \in N$ and $p \nmid j$, and (1), modulo p^e we obtain

$$P(n)=P(mp^e)=\sum_{d\mid mp^e}\mu(d)L_{\frac{m}{d}p^e}=\sum_{d\mid m}\mu(d)L_{\frac{m}{d}p^e}+\sum_{j\mid m}\mu(jp)L_{\frac{m}{j}p^{e-1}}$$
$$\equiv\sum_{d\mid m}\mu(d)L_{\frac{m}{d}p^{e-1}}-\sum_{j\mid m}\mu(j)L_{\frac{m}{j}p^{e-1}}\equiv 0 \pmod{p^e}.$$

Clearly, this proves the desired relation $P(n)\equiv 0 \pmod{n}$.

Modulo p^e we have

$$Q(n)=Q(mp^e)=\sum_{d\mid mp^e}\Phi(d)L_{\frac{m}{d}p^e}=\sum_{d\mid m}\Phi(d)L_{\frac{m}{d}p^e}+\sum_{\substack{d\mid mp^e\\ p\mid d}}\Phi(d)L_{\frac{m}{d}p^e}$$
$$\equiv\sum_{d\mid m}\Phi(d)L_{\frac{m}{d}p^{e-1}}+\sum_{s=1}^{e}\sum_{j\mid m}\Phi(jp^s)L_{\frac{m}{j}p^{e-s}} \pmod{p^e},$$

where we have used (1). Since $\Phi(jp^s)=(p^s-p^{s-1})\Phi(j)$ if $j,s\in N$ and $p\nmid j$, we obtain

$$\sum_{s=1}^{e}\sum_{j\mid m}\Phi(jp^s)L_{\frac{m}{j}p^{e-s}}=\sum_{s=1}^{e}(p^s-p^{s-1})\sum_{j\mid m}\Phi(j)L_{\frac{m}{j}p^{e-s}}$$
$$=\sum_{s=1}^{e}p^s\sum_{j\mid m}\Phi(j)L_{\frac{m}{j}p^{e-s}}-\sum_{t=0}^{e-1}p^t\sum_{j\mid m}\Phi(j)L_{\frac{m}{j}p^{e-t-1}}$$
$$\equiv\sum_{s=1}^{e-1}p^s\sum_{j\mid m}\Phi(j)L_{\frac{m}{j}p^{e-s}}-\sum_{t=0}^{e-1}p^t\sum_{j\mid m}\Phi(j)L_{\frac{m}{j}p^{e-t-1}}$$
$$\equiv\sum_{s=1}^{e-1}p^s\sum_{j\mid m}\Phi(j)L_{\frac{m}{j}p^{e-s-1}}-\sum_{t=0}^{e-1}p^t\sum_{j\mid m}\Phi(j)L_{\frac{m}{j}p^{e-t-1}}$$
$$=-\sum_{j\mid m}\Phi(j)L_{\frac{m}{j}p^{e-1}} \pmod{p^e},$$

where we have used (1) again. It follows that $Q(n)\equiv 0 \pmod{p^e}$. Of course, this proves the desired result $Q(n)\equiv 0 \pmod{n}$.

Also solved by P. Haukkanen and the proposer.

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