RODRIGUES' FORMULAS FOR JACOBSTHAL-TYPE POLYNOMIALS

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1. INTRODUCTION

Motivation

Recently [2], some second-order differential properties of generalized Fibonacci polynomials and generalized Lucas polynomials were exhibited.

Here, we intend to

(i) obtain similar differential equations from a slightly different viewpoint in the more general context of the polynomials $W_n(x)$ and $\mathcal{W}_n(x)$ [3], and

(ii) discover analogous equations for Jacobsthal polynomials $J_n(x)$ and Jacobsthal-Lucas polynomials $j_n(x)$ [4], i.e., non-Fibonacci and non-Lucas polynomials.

Central to the process is the question:

Can we determine Rodrigues' formulas for $J_n(x)$ and $j_n(x)$ corresponding to those (in a somewhat different notation) for $U_n(x)$ and $V_n(x)$ in [2]?

Background

Essentially, the following basic material [3] is needed:

$$W_{n+2}(x) = p(x)W_{n+1}(x) + q(x)W_n(x), \quad W_0(x) = 0,\ W_1(x) = 1, \tag{1.1}$$

$$\mathcal{W}_{n+2}(x) = p(x)\mathcal{W}_{n+1}(x) + q(x)\mathcal{W}_n(x), \quad \mathcal{W}_0(x) = 2,\ \mathcal{W}_1(x) = p(x), \tag{1.2}$$

leading to (if we drop the functional notation)

$$W_n = \frac{\alpha^n - \beta^n}{\Delta}, \tag{1.3}$$

$$\mathcal{W}_n = \alpha^n + \beta^n, \tag{1.4}$$

where

$$\left.\begin{aligned} \alpha &= \tfrac{1}{2}\{p + \Delta\}, \\ \beta &= \tfrac{1}{2}\{p - \Delta\}, \\ \Delta &= \sqrt{p^2 + 4q} = \alpha - \beta. \end{aligned}\right\} \tag{1.5}$$

Differentiating once w.r.t. x gives

$$\Delta' = \frac{pp' + 2q'}{\Delta}. \tag{1.6}$$

Specialized cases of (1.1) and (1.2) are generalized the Fibonacci and Lucas polynomials $F_n = W_n$ and $L_n = \mathcal{W}_n$, for which $p = x, q = 1$, and the Jacobsthal and Jacobsthal-Lucas polynomials J_n and j_n, for which $p = 1, q = 2x$. (See [3] for other examples of "Fibonacci-type" polynomials, e.g., Pell, Chebyshev, and Fermat.)

Two dichotomous situations thus arise:

A. $q' = 0$ for "Fibonacci-type" polynomials like F_n and L_n;
B. $p' = 0$ for J_n and j_n.

Immediately from (1.6) we have

$$\Delta' = \begin{cases} \dfrac{pp'}{\Delta}, & (1.6A) \\ \dfrac{2q'}{\Delta}. & (1.6B) \end{cases}$$

Crucial to the theory are the derivatives [3]

$$\mathcal{W}_n' = \begin{cases} np'W_n & (q'=0), \\ nq'W_{n-1} & (p'=0), \end{cases} \tag{1.7}$$

so, in particular,

$$j_n' = 2nJ_{n-1}. \tag{1.8}$$

Finally, we record for later use the notation [2]

$$c_{n,0} = 2\frac{n!}{(2n)!} \quad (n \geq 0), \tag{1.9}$$

and

$$c_{n,r} = 2\frac{n!n(n+r)!}{(2n)!(n+r)(n-r)!} \quad (n \geq r \geq 1), \tag{1.10}$$

whence

$$c_{n,r+1} = (n^2 - r^2)c_{n,r} \quad (n \geq r+1 \geq 1). \tag{1.11}$$

Notation for Theorems: Letters F and $J(j)$ will be appended as subscripts to the Theorem number of theorems relating to Fibonacci-type polynomials and Jacobsthal-type polynomials, respectively. In this symbolism, we will have Theorem $1_F, \ldots,$ Theorem 3_J.

2. SOME BASIC DIFFERENTIAL EQUATIONS FOR RECURRENCES

A. Fibonacci-type Polynomials ($q' = 0$)

From (1.3)-(1.7), double differentiation of $\mathcal{W}_n$ leads to

$$\Delta^2 \mathcal{W}_n'' = n^2(p')^2 \mathcal{W}_n - np(p')^2 W_n$$

whence, with $\mathcal{W}_n = y$,

$$\Delta^2 y'' + pp'y' - (np')^2 y = 0. \tag{2.1}$$

Alternatively, if we follow the procedure in [2], while using our notation, then we arrive at (2.1) also, a process left to the reader.

Differentiating (2.1) r times in conjunction with Leibniz' rule, we deduce that $z = y^{(r)} = \mathcal{W}_n^{(r)}$ satisfies the differential equation

$$\Delta^2 z'' + (2r+1)pp'z' + (p')^2(r^2 - n^2)z = 0, \tag{2.2}$$

of which (2.1) is the special case when $r = 0$.

Illustrations of (2.1) are:

(i) the associated Morgan-Voyce polynomial $C_n = y$, for which $p = 2 + x, q = -1$, leading to [2]

$$x(x+4)y'' + (x+2)y' - n^2 y = 0;$$

(ii) the Chebyshev polynomial $T_n = y$, in which $p = 2x, q = -1\,(x = \cos\theta)$, yielding

$$(1-x^2)y'' - xy' + n^2 y = 0,$$

in conformity with [6, p. 260].

Starting now with the double differentiation of W_n in (1.3), we eventually arrive at the differential equation

$$\Delta^2 W_n'' + 3pp'W_n' - (p')^2(n^2-1)W_n = 0. \tag{2.3}$$

Compare this with (2.1). A quick check confirms that $r = 1$ in (2.2) does indeed give us (2.3), where we invoke (1.7) for $q' = 0$. Particular instances of (2.3) are

(a) the Morgan-Voyce polynomial B_n, for which $p = 2 + x, q = -1$, giving

$$x(x+4)B_n'' - 3(x+2)B_n' - (n^2-1)B_n = 0,$$

in conformity with [2, p. 455] on making the transformation $n \to n-1$ for our B_n;

(b) The Chebyshev polynomial S_n (in the notation of [2, p. 453]), where $p = 2x$, $q = -1$ ($x = \cos\theta$), for which

$$(1-x^2)S_n'' - 3xS_n' + (n^2-1)S_n = 0$$

as in [6, p. 260], n being replaced by $n-1$ for our S_n.

Now (1.7), where $q' = 0$, immediately shows that $\mathcal{W}_n^{(r)} = np'W_n^{(r-1)}$ $(r \geq 1)$, i.e.,

$$W_n^{(r-1)} = \frac{1}{np'}\mathcal{W}_n^{(r)}. \tag{2.4}$$

Hence, $W_n^{(r-1)}$ satisfies (2.2). Combining this with (2.2), we deduce that

Theorem $\boldsymbol{1_F}$: $W_n^{(r-1)}$ and $\mathcal{W}_n^{(r)}$ both satisfy (2.2).

Example **(*r* = 2, *n* = 4; *p* = 2*x*, *q* = 1, *Pell-type polynomials [3]*):** $P_4^{(1)} = (8x^3 + 4x)'$ and $Q_4^{(2)} = (16x^4 + 16x^2 + 2)''$ both satisfy

$$(x^2+1)z'' + 5xz' - 12z = 0.$$

Observe that (2.2) can be cast in the more general form (cf. [2]):

$$\left[\Delta^{2r+1}z'\right]' = (p')^2(n^2 - r^2)\Delta^{2r-1}z. \tag{2.5}$$

Following the technique in [2] and using (2.5), we may establish the results corresponding to equations (2.9)-(2.11) in [2], namely $\left(\text{with } D^{(r)} \equiv \frac{d^r}{dx^r}\right)$:

$$D\left[\Delta^{2r+1}D^{(n+r)}\Delta^{2n-1}\right] = (p')^2(n^2-r^2)\Delta^{2r-1}D^{(n+r-1)}\Delta^{2n-1}, \tag{2.6}$$

$$D\left[\Delta^{-2r-1}D^{(n-r-1)}\Delta^{2n-1}\right] = (p')^2(n^2-(r+1)^2)\Delta^{-2r-3}D^{(n-r-2)}\Delta^{2n-1}, \tag{2.7}$$

$$D\left[\Delta D^{(n+1)}\Delta^{2n+1}\right] = (p')^2(n+1)^2\Delta^{-1}D^{(n)}\Delta^{2n+1}. \tag{2.8}$$

B. Jacobsthal (≡ non-Fibonacci)-type Polynomials ($p' = 0$)

Trying to apply the method used in [2], or variations of it, to J_n and j_n is likely to lead to frustration.

Therefore, we abandon this approach and start afresh.

Differentiate twice in the pivotal relation (1.7) for $p' = 0$. Then

$$\Delta^2\mathcal{W}_n'' + (q')^2\mathcal{W}_n' - n(n-1)(q')^2\mathcal{W}_{n-2} = 0, \tag{2.9}$$

wherein the diminished subscript in the undifferentiated polynomial is particularly to be noted. [Check (2.9) when, for example, $j_4 = 8x^2+8x+1$, $j_6 = 16x^3+36x^2+12x+1$, for which $p=1$, $q=2x$, $\Delta^2 = 1+8x$.]

Continued differentiation with recourse to Leibniz' rule, as in [2], reveals the generalized form of (2.9) to be ($z_n = \mathcal{W}_n^{(r)}$)

$$\Delta^2 z_n'' + (4r+q')q'z_n' - n(n-1)(q')^2 z_{n-2} = 0. \tag{2.10}$$

Putting $r = 0$ in (2.10) obviously leads us back to (2.9).

Repeated differentiation in (1.3) next yields, with little difficulty,

$$\Delta^2 W_n'' + 3(q')^2 W_n' - n(n-1)(q')^2 W_{n-2} = 0. \tag{2.11}$$

Contrast this with (2.3). One may readily verify (2.11) for, say, $J_5 = 4x^2+6x+1$, $J_7 = 8x^3 + 24x^2+10x+1$.

Proceeding for the sake of interest to differentiate (2.11) may times, we eventually arrive at the generalization ($z_n = W_n^{(r-1)}$)

$$\Delta^2 z_n'' + (4r+q')q'z_n' - n(n-1)(q')^2 z_{n-2} = 0. \tag{2.12}$$

Substituting $r = 1$ clearly reproduces (2.11), since $q' = 2$.

Bearing in mind (1.7) with $p' = 0$ and (2.12), we conclude that

Theorem 1_J: $J_n^{(r-1)}$ and $j_n^{(r)}$ both satisfy (2.10).

Analogously to (2.5), we see that (2.10) may be reformulated as

$$\left[\Delta^{2r+1}z_n'\right]' = (q')^2 n(n-1)\Delta^{2r-1}z_{n-2}.$$

Corresponding to (2.6)-(2.8), we derive

$$D\left[\Delta^{2r+1}D^{(n+r)}\Delta^{2n-1}\right] = (q')^2 n(n-1)\Delta^{2r-1}D^{(n+r-3)}\Delta^{2n-1}, \tag{2.13}$$

$$D\left[\Delta^{-2r-1}D^{(n-r-1)}\Delta^{2n-1}\right] = (q')^2 n(n-1)\Delta^{-2r-3}D^{(n-r-4)}\Delta^{2n-1}, \tag{2.14}$$

$$D\left[\Delta D^{(n+1)}\Delta^{2n+1}\right] = (q')^2 n(n+1)\Delta^{-1}D^{(n-2)}\Delta^{2n+1}. \tag{2.15}$$

3. RODRIGUES' FORMULAS

Rodrigues' formulas for W_n, $\mathcal{W}_n$ (when $q'=0$) and for J_n, j_n (when $p'=0$) are now determined.

A. Case $q'=0$.

Procedures followed in [2] using (1.9) will largely be applied here.

Theorem 2_F:

(i) $W_n = \dfrac{nc_{n,0}}{(p')^{n-1}}\Delta^{-1}D^{(n-1)}\Delta^{2n-1}$;

(ii) $\mathcal{W}_n = \dfrac{c_{n,0}}{(p')^n}\Delta D^{(n)}\Delta^{2n-1}$.

Proof: Definitions (1.3) and (1.4) disclose that

$$\mathcal{W}_{n+1} = \frac{1}{2}\left[p\mathcal{W}_n + \Delta^2 W_n\right]. \tag{3.1}$$

Assuming (i), (ii) in Theorem 2_F, we then have, on simplifying,

$$\mathcal{W}_{n+1} = \frac{n!\,\Delta}{(2n)!(p')^n}\left[pD^{(n)}\Delta^{2n-1} + np'D^{(n-1)}\Delta^{2n-1}\right]. \tag{3.2}$$

But, by Leibniz' rule,

$$\begin{aligned} D^{(n+1)}\Delta^{2n+1} &= D^{(n)}\{(2n+1)pp'\Delta^{2n-1}\} \\ &= (2n+1)p'\{pD^{(n)}\Delta^{2n-1} + np'D^{(n-1)}\Delta^{2n-1}\}, \end{aligned} \tag{3.3}$$

since $p''=0$. Accordingly, (3.2), (3.3) yield

$$\mathcal{W}_{n+1} = \frac{2(n+1)!\,\Delta}{(2n+2)!(p')^{n+1}}D^{(n+1)}\Delta^{2n+1}$$

in conformity with Theorem 2_F(ii) and (1.9).

Furthermore, from (1.7),

$$\begin{aligned} W_{n+1} &= \frac{1}{(n+1)p'}\mathcal{W}'_{n+1} \\ &= \frac{1}{(n+1)p'}\frac{c_{n+1,0}}{(p')^{n+1}}D\left(\Delta D^{(n+1)}\Delta^{2n+1}\right) \quad \text{by Theorem } 2_F\text{(ii)} \\ &= \frac{2(n+1)}{(p')^n}c_{n+1,0}\Delta^{-1}D^{(n)}\Delta^{2n+1} \quad \text{by (2.8)} \end{aligned}$$

in agreement with Theorem 2_F(i). Consequently, Theorem 2_F is completely proved.

***Example* (Chebyshev polynomials [3], $p=2x$, $q=-1$):**

$$W_5 = 16x^4 - 12x^2 + 1 \;\; (= U_4 \text{ [5, p. 256]}),$$
$$\mathcal{W}_5 = 2(16x^5 - 20x^3 + 5x) \;\; (= 2T_5 \text{ [5, p. 256]}).$$

See also [7, p. 755]. Be it noted that in [6] the Rodrigues formulas for Chebyshev polynomials are given in terms of Gamma functions.

More generally,

Theorem 3_F***:***

(i) $W_n^{(r)} = \dfrac{c_{n,r+1}}{n(p')^{n-2r-1}} \Delta^{-2r-1} D^{(n-r-1)} \Delta^{2n-1}$;

(ii) $\mathcal{W}_n^{(r)} = \dfrac{c_{n,r}}{(p')^{n-2r}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1}$.

Proof:

(i) Induction on r is employed. The Theorem is true for $r=0$ [Theorem 2_F(i)] and may be verified for $r=1,2$. Assume it is true for $r=k$. Then

$$W_n^{(k+1)} = \frac{c_{n,k+1}}{n(p')^{n-2k-1}} D\left[\Delta^{-2k-1} D^{(n-k-1)} \Delta^{2n-1}\right] \quad \text{by Theorem } 3_F\text{(i)}$$

$$= \frac{c_{n,k+2}}{n(p')^{n-2(k+1)-1}} \left[\Delta^{-2(k+1)-1} D^{(n-(k+1)-1)} \Delta^{2n-1}\right] \quad \text{by (2.7)}$$

as expected. Thus, the Theorem is true for $r=k+1$. Hence, it is true for all r.

(ii)

$$\mathcal{W}_n^{(r)} = np' W_n^{(r-1)} \quad \text{by (1.7)}$$

$$= np' \frac{c_{n,r}}{n(p')^{n-2r+1}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1} \quad \text{by Theorem } 3_F\text{(i)}$$

$$= \frac{c_{n,r}}{(p')^{n-2r}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1}$$

as desired. Thus, Theorem 3_F is completely established.

Examples:

Chebyshev: $W_5^{(1)} = 8x(8x^2-3)$;

Fermat: $\mathcal{W}_4^{(2)} = 36(27x^2-4)$. (Here, $p=3x, q=-2$.)

B. Case $p'=0$.

Efforts to exploit the techniques of the theory when $q'=0$ to the related situation when $p'=0$ are doomed to disappointment, due mainly to the differing natures of Δ' in (1.6A) and (1.6B). A fresh approach is therefore necessary.

Computations rapidly show that, since $\Delta' = 2q'/\Delta$ (1.6B),

$$\begin{aligned} D^{(1)}\Delta^{2n-1} &= (2n-1)(2q')\Delta^{2n-3}, \\ D^{(2)}\Delta^{2n-1} &= (2n-1)(2n-3)(2q')^2\Delta^{2n-5}, \\ &\cdots \\ D^{(n-1)}\Delta^{2n-1} &= (2n-1)(2n-3)(2n-5)\cdot\cdots\cdot 3(2q')^{n-1}\Delta, \end{aligned} \tag{3.4}$$

whence

$$\begin{gathered}
\binom{n}{1}\frac{c_{n,0}}{(q')^{n-1}}\Delta^{-1}D^{(n-1)}\Delta^{2n-1}=\binom{n}{1},\\
\binom{n}{3}\frac{c_{n-2,0}}{(q')^{n-3}}\Delta^{-1}D^{(n-2)}\Delta^{2n-1}=\binom{n}{3}\Delta^{2},\\
\cdots\\
\begin{cases}\binom{n}{n-1}\frac{c_{2,0}}{(q')^{1}}\Delta^{-1}D^{(1)}\Delta^{2n-1}=\binom{n}{n-1}\Delta^{n-2}, & n \text{ even},\\ \binom{n}{n}\frac{c_{1,0}}{(q')^{1}}\Delta^{-1}D^{(1)}\Delta^{2n-1}=\binom{n}{n}\Delta^{n-1}, & n \text{ odd}.\end{cases}
\end{gathered}\tag{3.5}$$

Differentiating once more in (3.4) gives rise to

$$D^{(n)}\Delta^{2n-1}=(2n-1)(2n-3)(2n-5)\cdot\dots\cdot 3\cdot 1(2q')^{n}\Delta^{-1}.\tag{3.6}$$

Initially

$$D^{(0)}\Delta^{2n-1}=\Delta^{2n-1}.\tag{3.7}$$

Reassembling the ideas in (3.4), (3.5), and (3.6), we arrive at

$$\begin{gathered}
\binom{n}{0}\frac{c_{n,0}}{(q')^{n}}\Delta D^{(n)}\Delta^{2n-1}=\binom{n}{0},\\
\binom{n}{2}\frac{c_{n-2,0}}{(q')^{n-2}}\Delta D^{(n-1)}\Delta^{2n-1}=\binom{n}{2}\Delta^{2},\\
\cdots\\
\begin{cases}\binom{n}{n}\frac{c_{2,0}}{(q')^{0}}\Delta D^{(1)}\Delta^{2n-1}=\binom{n}{n}\Delta^{n}, & n \text{ even},\\ \binom{n}{n-1}\frac{c_{1,0}}{(q')^{0}}\Delta D^{(1)}\Delta^{2n-1}=\binom{n}{n-1}\Delta^{n-1}, & n \text{ odd}.\end{cases}
\end{gathered}\tag{3.8}$$

Because the left-hand forms in (3.5) and (3.8) resemble the Rodrigues formulas in Theorem 2_F, we feel justified to appropriate to them the name of *Rodrigues-type* expressions.

Now, $p=1$ and $q=2x$ in (1.1), (1.3), and (1.5) indicate that

$$\begin{aligned}
J_n&=\frac{(1+\Delta)^n-(1-\Delta)^n}{\Delta}\qquad(\Delta^2=1+8x)\\
&=\frac{1}{2^{n-1}}\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}\Delta^{2k}=\frac{1}{2^{n-1}}\left[\binom{n}{1}+\binom{n}{3}\Delta^2+\binom{n}{5}\Delta^4+\cdots+\begin{cases}\binom{n}{n-1}\Delta^{n-2}\\ \binom{n}{n}\Delta^{n-1}\end{cases}\right]\begin{cases}n \text{ even},\\ n \text{ odd},\end{cases}\\
&=\text{a sum of expressions of Rodrigues-type (3.5).}
\end{aligned}\tag{3.9}$$

Similarly, use of (1.2), (1.4), and (1.5) gives rise to

$$\begin{aligned}
j_n&=(1+\Delta)^n+(1-\Delta)^n=\frac{1}{2^{n-1}}\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\Delta^{2k}\\
&=\frac{1}{2^{n-1}}\left[\binom{n}{0}+\binom{n}{2}\Delta^2+\binom{n}{4}\Delta^4+\cdots+\begin{cases}\binom{n}{n}\Delta^{n}\\ \binom{n}{n-1}\Delta^{n-1}\end{cases}\right]\begin{cases}n \text{ even},\\ n \text{ odd},\end{cases}\\
&=\text{a sum of expressions of Rodrigues-type (3.8).}
\end{aligned}\tag{3.10}$$

Combining (3.9) and (3.10), we then conclude that

Theorem $\mathbf{2}_J$***:*** The Rodrigues formula analogs for J_n and j_n are given by (3.9) and (3.10).

Examples:

$$J_6 = \frac{1}{32}\left[\binom{6}{1}+\binom{6}{3}(1+8x)+\binom{6}{5}(1+8x)^2\right] = 1+8x+12x^2;$$

$$j_6 = \frac{1}{32}\left[\binom{6}{0}+\binom{6}{2}(1+8x)+\binom{6}{4}(1+8x)^2+\binom{6}{6}(1+8x)^3\right] = 1+12x+36x^2+16x^3.$$

Our last major program is to generalize Theorem 2_J. Recall, first, that $j_n^{(1)} = 2nJ_{n-1}$ (1.8). Elementary calculations involving (1.3) and (1.4) for J_n and j_n quickly tell us that

$$J_n^{(1)} = \frac{4}{\Delta^2}\left(\frac{n}{2}j_{n-1} - J_n\right). \tag{3.11}$$

Subsequent differentiation reveals that

$$J_n^{(2)} = \frac{4}{\Delta^2}[n(n-1)J_{n-2} - 3J_n^{(1)}],$$

$$J_n^{(3)} = \frac{4}{\Delta^2}[n(n-1)J_{n-2}^{(1)} - 5J_n^{(2)}],$$

$$J_n^{(4)} = \frac{4}{\Delta^2}[n(n-1)J_{n-2}^{(2)} - 7J_n^{(3)}],$$

and so on, suggesting the proposition that

Theorem $\mathbf{3}_J$***:*** $J_n^{(r)} = \dfrac{4}{\Delta^2}\{n(n-1)J_{n-2}^{(r-2)} - (2r-1)J_n^{(r-1)}\},\ r \ge 2.$

Proof: Induction on r demonstrates the validity of this assertion.

Successive differentiations in (1.8) then establish that

Theorem $\mathbf{3}_j$***:*** $j_n^{(r)} = 2nJ_{n-1}^{(r-1)} = \dfrac{8n}{\Delta^2}[(n-1)(n-2)J_{n-3}^{(r-3)} - (2r-3)J_{n-1}^{(r-2)}],\ r \ge 3.$

Example of Theorems $\mathbf{3}_J, \mathbf{3}_j$ ***(***$\boldsymbol{r=2, n=9}$***):***

$$J_9^{(2)} = \frac{12}{8x+1}[24J_7 - J_9^{(1)}] = 24(8x^2+20x+5) = \frac{1}{20}j_{10}^{(3)}.$$

Observations

(i) Summation procedures beginning with the definitions (1.3) and (1.4), and ending with (3.9) and (3.10), cannot be applied to the Fibonacci-type polynomials. This is because (3.9) and (3.10) are tied irrevocably to (3.5) and (3.8), both of which depend on $p' = 0$.

(ii) Corresponding to (3.1) for Fibonacci-type polynomials, for J_n and j_n we may derive

$$\mathcal{W}_{n+1} = \Delta^2 W_n - q\mathcal{W}_{n-1}. \tag{3.12}$$

Use of the Leibniz rule nexus in Theorem 2_F is impossible in the case of Jacobsthal-type polynomials J_n and j_n because of the diminished subscript for $\mathcal{W}$ on the right-hand side.

(iii) In (3.11), where $r=1$, the appearance of $\frac{n}{2}j_{n-1}$, which seems to break the pattern of the theorem, requires explanation. From (1.8),

$$\frac{n}{2}j_{n-1}=\frac{n}{2}\int_0^x \frac{d}{dx}j_{n-1}\,dx=\frac{n}{2}\cdot 2(n-1)\int_0^x J_{n-2}\,dx=n(n-1)J_{n-2}^{(-1)},$$

where integration is represented by the negative unit superscript. With this symbolism, the pattern in Theorem 3_J is valid for $r\ge 1$, and hence that in Theorem 3_j for $r\ge 2$.

4. ILLUSTRATION OF THEORY WHEN $n = 5$ (i.e., $2n - 1 = 9$)

Now

$$\begin{aligned}
&D^{(1)}\Delta^9=9(pp'+2q')\Delta^7, \quad \text{where } \Delta^2=p^2+4q \ \ (1.5),\\
&D^{(2)}\Delta^9=9\{7(pp'+2q')^2\Delta^5+(p')^2\Delta^7\},\\
&D^{(3)}\Delta^9=9\{7[5(pp'+2q')^3\Delta^3+3(p')^2(pp'+2q')\Delta^5]\},\\
&D^{(4)}\Delta^9=9\cdot 7\cdot 3[5(pp'+2q')^4\Delta+10(p')^2(pp'+2q')^2\Delta^3+(p')^4\Delta^5].
\end{aligned} \tag{I}$$

Therefore,

$$\frac{5\Delta^{-1}D^{(4)}\Delta^9}{9\cdot 7\cdot 5\cdot 3\cdot (p')^4}=\frac{1}{(p')^4}\left\{\binom{5}{4}(pp'+2q')^4+\binom{5}{2}(p')^2(pp'+2q')^2\Delta^2+\binom{5}{0}(p')^4\Delta^4\right\}. \tag{I'}$$

So, for $q'=0$, on simplifying,

$$\begin{aligned}
\frac{1}{2^4}(\text{R.H.S.})&=p^4+3p^2q+q^2=W_5,\\
&=\begin{cases}16x^4+12x^2+1 & \text{for the Pell polynomial } P_5\ [3]\text{: } p=2x,\ q=1,\\ (1+6x+4x^2 & \text{for the Jacobsthal polynomial } J_5\text{: } p=1,\ q=2x).\end{cases}
\end{aligned}$$

Differentiate (I) again to get

$$D^{(5)}\Delta^9=9\cdot 7\cdot 5\cdot 3\cdot\left[\frac{(pp'+2q')^5}{\Delta}+10(p')^2(pp'+2q')^3\Delta^2+5(p')^4(pp'+2q')\Delta^3\right]. \tag{II}$$

Then

$$\frac{\Delta D^{(5)}\Delta^9}{9\cdot 7\cdot 5\cdot 3\cdot 1\cdot (p')^5}=\frac{1}{(p')^5}\left\{\binom{5}{5}(pp'+2q')^5+\binom{5}{3}(p')^2(pp'+2q')^3\Delta^2+\binom{5}{1}(p')^4(pp'+2q')\Delta^4\right\}, \tag{II'}$$

whence, for $q'=0$,

$$\begin{aligned}
\frac{1}{2^4}(\text{R.H.S.})&=p^5+5p^3q+5pq^2=\mathcal{W}_5,\\
&=\begin{cases}32x^5+40x^3+10x & \text{for Pell-Lucas polynomials [3]},\\ (1+10x+20x^2 & \text{for Jacobsthal-Lucas polynomials}).\end{cases}
\end{aligned}$$

On the other hand, when $p'=0$, we obtain the results (3.4)-(3.8) and hence (3.9) and (3.10).

Notice, particularly, that the general expressions for W_5 and $\mathcal{W}_5$ above are valid for both Fibonacci-type and Jacobsthal-type polynomials, even though $q'=0$.

This is because the binomial coefficients associated with the powers of Δ *in (I') and (II') are the same as those in (3.9) and (3.10), since* $\binom{n}{m} = \binom{n}{n-m}$.

Expressions for W_n and $\mathcal{W}_n$ may be sighted in [5] in a notation slightly varied from that used here.

5. CONCLUSION

While the author of [2] evidently did not consider this theory as applying to Jacobsthal-type polynomials, one observes that if his numerical parameter q in [2, eqns. (1.1), (1.2)] is allowed to be functional $q(x) = -2x$ with accompanying change in his x and p, then J_n and j_n can be incorporated into his system. For example, his U_5 ([2, eqn. (1.12)] reduces to $1+6x+4x^2 = J_5$.

So we come to our rest, having achieved the objectives (i) and (ii) in Section 1 which motivated our undertaking. Many facets of the work were revealed with others to be investigated. The unexpected complications in the patterns of behavior of J_n and j_n (and W_n and $\mathcal{W}_n$) have added zest to the hunt.

Questions: Does there exist a general formula for the coefficients of the Jacobsthal-type polynomials in terms of the Gamma function in the sense of [1, Table 22.3]? If so, is it attainable using the techniques of this paper? Can, further, the theory be extended to the situation when both $p(x)$ and $q(x)$ are linear polynomials?

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