ON A CLASS OF GENERALIZED POLYNOMIALS

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(Submitted February 1996-Final Revision June 1996)

1. INTRODUCTION

In a series of articles [1]-[3], André-Jeannin has recently defined the polynomials $U_n(p,q;x)$ and $V_n(p,q;x)$ by the recurrence relations (1) and (2), and has studied some of the combinatorial properties of the coefficients of U_n and V_n as well as some of the differential properties of these polynomials.

$$U_n = (x+p)U_{n-1} - qU_{n-2} \ (n \geq 2),\ U_0 = 0, U_1 = 1 \tag{1}$$

and

$$V_n = (x+p)V_{n-1} - qV_{n-2} \ (n \geq 2),\ V_0 = 2, V_1 = x+p. \tag{2}$$

The parameters p and q as well as the variable x are real numbers. If α and β are defined by

$$\alpha + \beta = x + p,\ \alpha\beta = q, \tag{3}$$

then it is well known that [5]

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}}, \tag{4a}$$

and

$$V_n = \alpha^n + \beta^n, \tag{4b}$$

where

$$\Delta = (x+p)^2 - 4q. \tag{5}$$

The purpose of this article is to introduce and study some of the properties of the generalized polynomial $W_n(p,q;x)$ defined by

$$W_n = (x+p)W_{n-1} - qW_{n-2} \ (n \geq 2), \tag{6}$$

where W_0 and W_1 are specified, as well as those of two other polynomials $u_n(p,q;x)$ and $v_n(p,q;x)$ that are very closely associated with U_n and V_n. We shall define these polynomials $u_n(p,q;x)$ and $v_n(p,q;x)$ to be

$$u_n = (x+p)u_{n-1} - qu_{n-2} \ (n \geq 2),\ u_0 = 1, u_1 = x + p - \sqrt{q} \tag{7}$$

and

$$v_n = (x+p)v_{n-1} - qv_{n-2} \ (n \geq 2),\ v_0 = 1, v_1 = x + p + \sqrt{q}. \tag{8}$$

2. SOME BASIC RELATIONS AMONG U_n, V_n, u_n AND v_n

Using the well-known properties of $W_n(a,b,p,q)$ introduced by Horadam [5], we may derive a number of relations between U_n and V_n. However, we shall not do so except to list a few of the important ones that will be required for the remainder of this article. It is easy to show that W_n as defined by (6) may be evaluated using the relation [5],

$$W_n = W_1 U_n - qW_0 U_{n-1} \ (n \geq 1). \tag{9}$$

From (9) we can immediately derive the following relations:

$$V_n = U_{n+1} - qU_{n-1}, \tag{10}$$

$$u_n = U_{n+1} - \sqrt{q}\, U_n, \tag{11}$$

$$v_n = U_{n+1} + \sqrt{q}\, U_n, \tag{12}$$

$$V_n = u_n + \sqrt{q}\, u_{n-1} = v_n - \sqrt{q}\, v_{n-1}. \tag{13}$$

From the results in [5], we may also derive the following "Simson" formulas:

$$U_{n+1}U_{n-1} - U_n^2 = -q^{n-1}, \tag{14a}$$

$$V_{n+1}V_{n-1} - V_n^2 = q^{n-1}\Delta, \tag{14b}$$

$$u_{n+1}u_{n-1} - u_n^2 = q^{n-1/2}\Delta_u, \tag{14c}$$

$$v_{n+1}v_{n-1} - v_n^2 = -q^{n-1/2}\Delta_v, \tag{14d}$$

where

$$\Delta_u = x + p - 2\sqrt{q}, \tag{15a}$$

$$\Delta_v = x + p + 2\sqrt{q}, \tag{15b}$$

$$\Delta = \Delta_u \Delta_v. \tag{15c}$$

From (14a-14d), we have the interesting result that

$$q(U_{n+1}U_{n-1} - U_n^2)(V_{n+1}V_{n-1} - V_n^2) = (u_{n+1}u_{n-1} - u_n^2)(v_{n+1}v_{n-1} - v_n^2) = -q^{2n-1}\Delta. \tag{16}$$

3. ZEROS AND ORTHOGONALITY PROPERTY OF U_n, V_n, u_n, AND v_n

André-Jeannin ([1], [2]) has shown that

$$U_n = q^{(n-1)/2}\frac{\sin n\theta}{\sin\theta} \tag{17a}$$

and

$$V_n = 2q^{n/2}\cos n\theta, \tag{17b}$$

where $\cos\theta = (x+p)/2\sqrt{q}$. Hence, from (11) and (17a), we get

$$u_n = q^{n/2}\frac{\cos(2n+1)\theta/2}{\cos\theta/2}. \tag{17c}$$

Similarly, from (12) and (17a), we have

$$v_n = q^{n/2}\frac{\sin(2n+1)\theta/2}{\sin\theta/2}. \tag{17d}$$

Hence, the zeros of U_n, V_n, u_n, and v_n are given by

$$U_n:\ x_k = -p + 2\sqrt{q}\cos\left(\frac{k}{n}\cdot\pi\right), \qquad k = 1, 2, \ldots, n-1, \tag{18a}$$

$$V_n:\ x_k = -p + 2\sqrt{q}\cos\left(\frac{2k-1}{2n}\cdot\pi\right), \quad k = 1, 2, \ldots, n, \tag{18b}$$

$$u_n:\ x_k = -p + 2\sqrt{q}\cos\left(\frac{2k-1}{2n+1}\cdot\pi\right), \quad k = 1, 2, \ldots, n, \tag{18c}$$

$$v_n:\ x_k = -p + 2\sqrt{q}\cos\left(\frac{2k}{2n+1}\cdot\pi\right), \quad k = 1, 2, \ldots, n. \tag{18d}$$

Of these, André-Jeannin ([1], [2]) has given the zeros for U_n and V_n. It should be observed that, if $p=2$ and $q=1$, then the above results correspond to the already known results for the zeros of $B_n(x)$, $C_n(x)$, $b_n(x)$, and $c_n(x)$ (see [6], [7], [4]).

André-Jeannin ([1], [2]) has shown further that U_n and V_n are orthogonal over the interval $(-p-2\sqrt{q},\ -p+2\sqrt{q})$ with respect to the weight functions $w_U(x)=\sqrt{-\Delta}$ and $w_V(x)=1/w_U(x)$, respectively. Using expressions (17c) and (17d), we may easily prove that u_n and v_n are also orthogonal over the same interval, but with respect to the weight functions $w_u(x)=\sqrt{-\Delta_u/\Delta_v}$ and $w_v(x)=1/w_u(x)$, respectively.

4. Q-MATRIX AND FORMULAS FOR W_{nk-1}, W_{nk} AND W_{nk+1}

If we define the generating matrix Q to be

$$Q=\begin{bmatrix} x+p & -q \\ 1 & 0 \end{bmatrix}, \tag{19}$$

then it is straightforward to show by induction that

$$P = Q^k = \begin{bmatrix} U_{k+1} & -qU_k \\ U_k & -qU_{k-1} \end{bmatrix}. \tag{20}$$

The characteristic equation of P is given by

$$\lambda^2 - (U_{k+1} - qU_{k-1})\lambda + q(U_k^2 - U_{k+1}U_{k-1}) = 0.$$

Using relations (10) and (14a), we may reduce the above equation to

$$\lambda^2 - V_k\lambda + q^k = 0.$$

Hence, by the Cayley-Hamilton theorem, we have

$$P^2 = V_kP - q^kI. \tag{21}$$

Starting with (21), we may easily show by induction that

$$P^n(x) = \lambda_n(x)P(x) - q^k\lambda_{n-1}(x)I, \tag{22}$$

where $\lambda_n(x)$ satisfies the recurrence relation

$$\lambda_n(x) = V_k(x)\lambda_{n-1}(x) - q^k\lambda_{n-2}(x) \;\; (n \geq 2), \;\; \lambda_0 = 0, \lambda_1 = 1. \tag{23}$$

Hence, from (20) and (22), we have

$$Q^{nk}(x) = \lambda_n(x)Q^k(x) - q^k\lambda_{n-1}(x)I. \tag{24}$$

Therefore, we have

$$U_{nk}(x) = \lambda_n(x)U_k(x), \tag{25a}$$

$$U_{nk+1}(x) = \lambda_n(x)U_{k+1}(x) - q^k\lambda_{n-1}(x), \tag{25b}$$

and

$$U_{nk-1}(x) = \lambda_n(x)U_{k-1}(x) + q^{k-1}\lambda_{n-1}(x). \tag{25c}$$

We now derive similar results for the polynomial W, and thus for the polynomials V, u, and v. Consider the matrix

$$R = \begin{bmatrix} W_{nk+1} & -qW_{nk} \\ W_{nk} & -qW_{nk-1} \end{bmatrix}.$$

Using relation (9), we may rewrite R as

$$\begin{aligned} R &= W_1\begin{bmatrix} U_{nk+1} & -qU_{nk} \\ U_{nk} & -qU_{nk-1} \end{bmatrix} - qW_0\begin{bmatrix} U_{nk} & -qU_{nk-1} \\ U_{nk-1} & -qU_{nk-2} \end{bmatrix} \\ &= W_1Q^{nk} - qW_0Q^{nk-1}, \text{ using (20)}, \\ &= Q^{nk}(W_1I - qW_0Q^{-1}). \end{aligned}$$

Hence,

$$\begin{bmatrix} W_{nk+1} & -qW_{nk} \\ W_{nk} & -qW_{nk-1} \end{bmatrix} = \begin{bmatrix} U_{nk+1} & -qU_{nk} \\ U_{nk} & -qU_{nk-1} \end{bmatrix}\begin{bmatrix} W_1 & -qW_0 \\ W_0 & W_1 - (x+p)W_0 \end{bmatrix}.$$

From the above identity, we may derive the following relations after some manipulations using (9) and (25a-25c):

$$W_{nk} = \lambda_n W_k - q^k W_0 \lambda_{n-1}, \tag{26a}$$

$$W_{nk+1} = \lambda_n W_{k+1} - q^k W_1 \lambda_{n-1}, \tag{26b}$$

$$W_{nk-1} = \lambda_n W_{k-1} + q^{k-1}\lambda_{n-1}\{W_1 - (x+p)W_0\}. \tag{26c}$$

Using appropriate values for W_0 and W_1 in (26a-26c), we may now derive the following relations for the polynomials V, u, and v:

$$V_{nk} = \lambda_n V_k - 2q^k\lambda_{n-1}, \tag{27a}$$

$$V_{nk+1} = \lambda_n V_{k+1} - q^k(x+p)\lambda_{n-1}, \tag{27b}$$

$$V_{nk-1} = \lambda_n V_{k-1} - q^{k-1}(x+p)\lambda_{n-1}; \tag{27c}$$

$$u_{nk} = \lambda_n u_k - q^k\lambda_{n-1}, \tag{28a}$$

$$u_{nk+1} = \lambda_n u_{k+1} - q^k(x+p-\sqrt{q})\lambda_{n-1}, \tag{28b}$$

$$u_{nk-1} = \lambda_n u_{k-1} - q^{k-1/2}\lambda_{n-1}; \tag{28c}$$

$$v_{nk} = \lambda_n v_k - q^k \lambda_{n-1}, \tag{29a}$$

$$v_{nk+1} = \lambda_n v_{k+1} - q^k (x + p + \sqrt{q}) \lambda_{n-1}, \tag{29b}$$

$$v_{nk-1} = \lambda_n v_{k-1} + q^{k-1/2} \lambda_{n-1}. \tag{29c}$$

It is clear from (23) that, if $V_k \mid \lambda_{n-2}$, then $V_k \mid \lambda_n$ also. However, $V_k \mid \lambda_2$ since $\lambda_2 = V_k$. Hence, by induction, it follows that $V_k \mid \lambda_n$ when n is even. Thus, we see from (25a) that $V_k \mid U_{kn}$ for even n, while $U_k \mid U_{kn}$ for all n. Further, we see from (27a) that $V_k \mid V_{kn}$ for odd n. Thus, we have the following results:

$$U_k \mid U_{kn} \quad \text{for all } n; \tag{30a}$$

$$V_k \mid U_{kn} \quad \text{for even } n; \tag{30b}$$

$$V_k \mid V_{kn} \quad \text{for odd } n. \tag{30c}$$

It is evident that similar results hold for Fibonacci and Lucas polynomials, Pell and Pell-Lucas polynomials, etc., since these polynomials are special cases of U_n and V_n. In particular, for the Fibonacci, Lucas, Pell, and Pell-Lucas numbers F_n, L_n, P_n, and Q_n, we obtain from (30) the already known results:

$$F_k \mid F_{kn}, \quad P_k \mid P_{kn}, \quad \text{for all } n; \tag{31a}$$

$$L_k \mid F_{kn}, \quad Q_k \mid P_{kn}, \quad \text{for even } n; \tag{31b}$$

$$L_k \mid L_{kn}, \quad Q_k \mid P_{kn}, \quad \text{for odd } n. \tag{31c}$$

5. SPECIAL CASE WHEN $q = 1$

This corresponds to a modified version of the Morgan-Voyce polynomials, where $x+2$ is replaced by $x+p$ in the difference equations. We shall denote the modified Morgan-Voyce polynomials by $\widetilde{B}_n(x)$, $\widetilde{b}_n(x)$, $\widetilde{C}_n(x)$, and $\widetilde{c}_n(x)$, where

$$\widetilde{B}_n(x) = U_{n+1}(p, 1; x), \tag{32a}$$

$$\widetilde{C}_n(x) = V_n(p, 1; x), \tag{32b}$$

$$\widetilde{b}_n(x) = u_n(p, 1; x), \tag{32c}$$

$$\widetilde{c}_n(x) = v_n(p, 1; x). \tag{32d}$$

Hence, from (14a-14d), we have the "Simson" formulas:

$$\widetilde{B}_{n+1}\widetilde{B}_{n-1} - \widetilde{B}_n^2 = -1; \tag{33a}$$

$$\widetilde{C}_{n+1}\widetilde{C}_{n-1} - \widetilde{C}_n^2 = (x+p)^2 - 4 = \Delta = \Delta_b \Delta_c; \tag{33b}$$

$$\widetilde{b}_{n+1}\widetilde{b}_{n-1} - \widetilde{b}_n^2 = x + p - 2 = \Delta_b; \tag{33c}$$

$$\widetilde{c}_{n+1}\widetilde{c}_{n-1} - \widetilde{c}_n^2 = -(x+p+2) = -\Delta_c. \tag{33d}$$

André-Jeannin [3] has shown that $\widetilde{B}_n^{(k)}(x)$ and $\widetilde{C}_n^{(k)}(x)$, $k = 0, 1, 2, \ldots$, where k stands for the k^{th} derivative, satisfy the following second-order differential equations:

$$\widetilde{B}_n^{(k)}(x): \quad \Delta y'' + (2k+3)(x+p)y' + \{(k+1)^2 - (n+1)^2\}y = 0, \tag{34a}$$

$$\widetilde{C}_n^{(k)}(x): \quad \Delta y'' + (2k+1)(x+p)y' + (k^2 - n^2)y = 0, \tag{34b}$$

where

$$\Delta = (x+p)^2 - 4. \tag{34c}$$

We will now derive similar results for $\widetilde{b}_n^{(k)}(x)$ and $\widetilde{c}_n^{(k)}(x)$. It is already known (see [6]) that $b_n(x)$ satisfies the differential equation

$$x(x+4)b_n''(x) + 2(x+1)b_n'(x) - n(n+1)b_n(x) = 0. \tag{35}$$

Changing x to $x+p-2$ and noting that $\widetilde{b}_n(x) = b_n(x+p-2)$, we find that equation (35) reduces to

$$\Delta \widetilde{b}_n''(x) + 2(x+p-1)\widetilde{b}_n'(x) - n(n+1)\widetilde{b}_n(x) = 0, \tag{36}$$

where Δ is given by (34c). Differentiating (36) k times and using the Leibniz rule, we can show that $\widetilde{b}_n^{(k)}(x)$ satisfies the differential equation

$$\widetilde{b}_n^{(k)}(x): \ \Delta y'' + 2\{(k+1)(x+p) - 1\}y' + \{k(k+1) - n(n+1)\}y = 0. \tag{37a}$$

Similarly, we can show that $\widetilde{c}_n^{(k)}(x)$ satisfies the equation

$$\widetilde{c}_n^{(k)}(x): \ \Delta y'' + 2\{(k+1)(x+p) + 1\}y' + \{k(k+1) - n(n+1)\}y = 0. \tag{37b}$$

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AMS Classification Numbers: 11B39, 33C25

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