ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-536 ***Proposed by Paul S. Bruckman, Highwood, IL***

Given an odd prime p, integers n and r with $n \geq 1$, let $m = 2[\frac{1}{2}n] - 1$,

$$S_{n,r,p} = \sum_{k=1}^{p-1} F_m^k \cdot \frac{F_{nk+r}}{k}, \quad T_{n,r,p} = \sum_{k=1}^{p-1} F_m^k \cdot \frac{L_{nk+r}}{k}.$$

Prove the following congruences:

$$\text{(a)} \quad S_{n,r,p} \equiv \frac{F_n^p F_{mp+r} - F_m^p F_{np+r} + F_r}{p} \pmod{p};$$

$$\text{(b)} \quad T_{n,r,p} \equiv \frac{F_n^p L_{mp+r} - F_m^p L_{np+r} + L_r}{p} \pmod{p}.$$

H-537 ***Proposed by Stanley Rabinowitz, Westford, MA***

Let $\langle w_n \rangle$ be any sequence satisfying the recurrence

$$w_{n+2} = Pw_{n+1} - Qw_n.$$

Let $e = w_0 w_2 - w_1^2$ and assume $e \neq 0$ and $Q \neq 0$.

Computer experiments suggest the following formula, where k is an integer larger than 1:

$$w_{kn} = \frac{1}{e^{k-1}} \sum_{i=0}^{k} c_{k-i} \binom{k}{i} (-1)^i w_n^i w_{n+1}^{k-i},$$

where

$$c_i = \sum_{j=0}^{k-2} \binom{k-2}{j} (-Qw_0)^j w_1^{k-2-j} w_{i-j}.$$

Prove or disprove this conjecture.

H-538 ***Proposed by Paul S. Bruckman, Highwood, IL***

Define the sequence of integers $(B_k)_{k \geq 0}$ by the generating function:

$$(1-x)^{-1}(1+x)^{-\frac{1}{2}} = \sum_{k\geq 0} B_k \frac{\left(\frac{1}{2}x\right)^k}{k!}, \quad |x|<1 \text{ (see [1]).}$$

Show that

$$\sum_{k\geq 0} B_k^2 \cdot \frac{1}{(2k+2)!} = \frac{\pi^2}{8} - \frac{1}{4}\log^2 u, \text{ where } u = 1+\sqrt{2}.$$

Reference

1. P. S. Bruckman. "An Interesting Sequence of Numbers Derived from Various Generating Functions." *The Fibonacci Quarterly* **10.2** (1972):169-81.

SOLUTIONS

Find Your Identity

H-518 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 34, no. 5, November 1996)***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that, for all complex numbers x and y and all positive integers n,

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)F_k(y) = (x-y)^{n-1}F_n\left(\frac{xy+4}{x-y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$\sum_{\substack{k=0\\ 5\nmid 2n-k-1}}^{2n-1}(-1)^{[(2n-k+1)/5]}\binom{4n-2}{k} = 5^{n-1}L_{2n-1}; \tag{2}$$

$$\sum_{\substack{k=0\\ 5\nmid 2n-k}}^{2n}(-1)^{[(2n-k+2)/5]}\binom{4n}{k} = 5^{n}F_{2n}; \tag{3}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{3k}P_k = 2^nF_n(6), \text{ where } P_k = F_k(2) \text{ is the } k^{\text{th}} \text{ Pell number;} \tag{4}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)F_k(x+1) = F_n(x^2+x+4); \tag{5}$$

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(x)F_k(4/x) = \frac{1-(-1)^n}{2}\left(\frac{x^2+4}{x}\right)^{n-1}, \quad x\neq 0; \tag{6}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)^2 = (x^2+4)^{n-1}; \tag{7}$$

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(x)^2 = \frac{4^n-(-x^2)^n}{4+x^2}; \tag{8}$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{2n}{n-2k-1}F_{2k+1}(x) = x^{n-1}F_n(4/x). \tag{9}$$

The latter equation is the one given in H-500. **Hint:** Deduce (1) from the main identity of H-492.

Solution by the proposer

Proof of (1): From H-492, we know that

$$\sum_{k=0}^{[n/2]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=z^{n-1}F_n(xy/z),$$

where $z=\sqrt{x^2+y^2+4}$. Replacing n by $2n$ and substituting k by $n-k$ gives

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{2k}(x)F_{2k}(y)=z^{2n-1}F_{2n}(xy/z).$$

Using $F_{2k}(x)=i^{1-k}xF_k(i(x^2+2))$, $i=\sqrt{(-1)}$, we get

$$xy\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(i(x^2+2))\,F_k(i(y^2+2))=i^{1-n}xyz^{2n-2}F_n(i((xy/z)^2+2)).$$

Now, we replace x by $i\sqrt{2+ix}$ and y by $i\sqrt{2-iy}$, so that z becomes $\sqrt{i(y-x)}$. Then, using $(-1)^{k+1}F_k(-y)=F_k(y)$ and some elementary calculations, we obtain (1).

Proof of (2) and (3): Let $x=i\alpha$ and $y=i\beta$. In [1] it was shown that

$$F_k(i\alpha)F_k(i\beta)=\begin{cases}(-1)^{[(k+2)/5]} & \text{if } 5\nmid k,\\ 0 & \text{if } 5\mid k,\end{cases}$$

so that by (1),

$$\sum_{\substack{k=0\\5\nmid k}}^{n}(-1)^{[(k+2)/5]}\binom{2n}{n-k}=(i\sqrt{5})^{n-1}F_n(-i\sqrt{5}).$$

Replacing n by $2n-1$, using $F_{2n-1}(-i\sqrt{5})=(-1)^{n-1}L_{2n-1}$, and reindexing k by $2n-k-1$, we find (2).

Replacing n by $2n$, using $iF_{2n}(-i\sqrt{5})=(-1)^{n-1}\sqrt{5}F_{2n}$, and substituting k by $2n-k$ gives (3).

Proof of (4): This follows from (1) by taking $x=4$, $y=2$, and using $F_k(4)=F_{3k}/2$.

Proof of (5): Take $y=x+1$. We note the particular case,

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_kP_k=F_n(6),$$

obtained when $x=1$.

Proof of (6): Take $y=-4/x$, use $F_k(-4/x)=(-1)^{k+1}F_k(4/x)$ and $F_n(0)=(1-(-1)^n)/2$. Then, with $x=1$, we obtain

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_kF_{3k}=(1-(-1)^n)5^{n-1}.$$

Proof of (7): Take $y=x$.

Proof of (8): Take $y=-x$ and use

$$(2x)^{n-1}F_n\left(\frac{4-x^2}{2x}\right)=\frac{4^n-(-x^2)^n}{4+x^2},$$

which easily follows from the well-known Binet form of the Fibonacci polynomials. With $x=2$, we get

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}P_k^2=(1-(-1)^n)2^{2n-3}.$$

Proof of (9): Take $y=0$.

Reference

1. N. Jensen. "Solution of H-492." *The Fibonacci Quarterly* **34.1** (1996):91-96.

Also solved by P. Bruckman.

Squares among US

H-520 ***Proposed by Andrej Dujella, University of Zagreb, Croatia (Vol. 34, no. 5, November 1996)***

Let n be an integer. Prove that there exists an infinite set $D\subseteq \mathbf{N}$ with the property that, for all $c, d\in D$, the integer $cd+n$ is not squarefree.

Solution by David Terr, University of California at Berkeley, CA

We claim that, for all n, an arithmetic sequence

$$D=\{kp^2+a\,|\,k\in\mathbf{N}\}$$

satisfying the above property exists, where p is a prime and $a<p^2/2$ is a nonnegative integer. If $4|n$, we may choose $p=2$ and $a=0$. If $n\equiv 3 \pmod 4$, we may choose $p=2$ and $a=1$. Finally, if $n\equiv 1$ or $2 \pmod 4$, we choose p to be an odd prime such that $(\frac{-n}{p})=1$ and find a nonnegative integer $a<p^2/2$ such that $a^2\equiv -n(p^2)$. By Hensel's lemma, such an a exists and is unique.

To see that D satisfies the above property, first consider the case in which $4|n$. In this case, $D=\{4k\,|\,k\in\mathbf{N}\}$, so if $c, d\in D$, we have $c=4k$ and $d=4l$ for some $k, l\in\mathbf{N}$, whence $cd+n=16kl+n$, which is divisible by 4 and, thus, not squarefree.

Next, consider the case in which $n\equiv 3 \pmod 4$. In this case, $D=\{4k+1\,|\,k\in\mathbf{N}\}$, so if $c, d\in D$, we have $c=4k+1$ and $d=4l+1$ for some $k, l\in\mathbf{N}$, whence $cd+n=16kl+4(k+l)+1+n$, which is again divisible by 4 and, thus, not squarefree.

Finally, consider the case in which $n\equiv 1$ or $2 \pmod 4$. In this case, $D=\{kp^2+a\,|\,k\in\mathbf{N}\}$ for some odd prime p and some nonnegative integer $a<p^2/2$ such that $p^2\,|\,(a^2+n)$. If $c, d\in D$, we have $c=kp^2+a$ and $d=lp^2+a$ for some $k, l\in\mathbf{N}$, whence $cd+n=klp^4+a(k+l)p^2+a^2+n$, which is divisible by p^2 and, thus, not squarefree. □

The following table lists the values of p and a found by this method for $|n|\leq 10$.

n	p	a	n	p	a
−10	3	1	0	2	0
−9	2	1	1	5	7
−8	2	0	2	3	4
−7	3	4	3	2	1
−6	5	9	4	2	0
−5	2	1	5	3	2
−4	2	0	6	5	12
−3	11	27	7	2	1
−2	7	10	8	2	0
−1	2	1	9	5	4
			10	7	23

Also solved by B. Beasley, P. Bruckman, and the proposer.

Zeroing In

H-521 ***Proposed by Paul S. Bruckman, Highland, IL***
(Vol. 35, no. 1, February 1997)

Let ρ denote any zero of the Riemann Zeta Function $\zeta(z)$ lying in the strip

$$S = \{z \in C: 0 < \operatorname{Re}(z) < 1\}.$$

Prove the following:

(1) $\sum_{\rho \in S}\left(\rho - \frac{1}{2}\right)^{-1} = 0;$

(2) $\sum_{\rho \in S}\rho^{-1} = 1 + \frac{1}{2}\gamma - \frac{1}{2}\log 4\pi$, where γ is Euler's Constant.

Solution by Kee-Wai Lau, Hong Kong

Proof of (1): It is well known that the zeros are in conjugate pairs. They either lie on the line $\operatorname{Re} z = \frac{1}{2}$ or occur in pairs symmetrical about this line. If $\operatorname{Re}\rho = \frac{1}{2}$, we have

$$\frac{1}{\rho - \frac{1}{2}} + \frac{1}{\overline{\rho} - \frac{1}{2}} = 0.$$

If $\operatorname{Re}\rho \neq \frac{1}{2}$, then ρ is a zero if and only if $\overline{\rho}$, $1-\rho$, and $1-\overline{\rho}$ are zeros, and we have

$$\frac{1}{\rho - \frac{1}{2}} + \frac{1}{\overline{\rho} - \frac{1}{2}} + \frac{1}{(1-\rho) - \frac{1}{2}} + \frac{1}{(1-\overline{\rho}) - \frac{1}{2}} = 0. \quad \square$$

Proof of (2): It is known (see [1], Formula 2.12.7, p. 31) that

$$\frac{\zeta'(z)}{\zeta(z)\gamma} = \log 2\pi - 1 - \frac{1}{2}\gamma - \frac{1}{z-1} - \frac{1}{2}\frac{\Gamma'((z/2)+1)}{\Gamma((z/2)+1)} + \sum_{\rho \in S}\left(\frac{1}{z-\rho} + \frac{1}{\rho}\right), \qquad (*)$$

where Γ is the Gamma function.

It is also known (see [1], p. 20) that

$$-\frac{\zeta'(1-z)}{\zeta(1-z)} = -\log 2\pi - \frac{1}{2}\pi \tan\frac{1}{2}z\pi + \frac{\Gamma'(z)}{\Gamma(z)} + \frac{\zeta'(z)}{\zeta(z)}. \tag{$**$}$$

By substituting $z=\frac{1}{2}$ into $(*)$ and $(**)$ and making use of (1) we obtain, after some algebra,

$$\sum_{\rho\in S}\frac{1}{\rho} = \frac{1}{2}\frac{\Gamma'(5/4)}{\Gamma(5/4)} - 1 + \frac{1}{2}\gamma - \frac{1}{2}\log 2\pi + \frac{\pi}{4} - \frac{1}{2}\frac{\Gamma'(1/2)}{\Gamma(1/2)}.$$

Since

$$\frac{\Gamma'(1/2)}{\Gamma(1/2)} = -\gamma - 2\log 2,$$

in order to prove (2) it remains to show that

$$\frac{\Gamma'(5/4)}{\Gamma(5/4)} = -\gamma - 3\log 2 - \frac{\pi}{2} + 4.$$

In fact, by substituting $z=\frac{1}{4}$ into the duplication formula

$$\frac{\Gamma'(2z)}{\Gamma(2z)} = \frac{1}{2}\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{2}\frac{\Gamma'(z+(1/2))}{\Gamma(z+(1/2))} + \log 2$$

and into the reflection formula

$$\frac{\Gamma'(1-z)}{\Gamma(1-z)} = \frac{\Gamma'(z)}{\Gamma(z)} + \pi\cot\pi z,$$

we easily obtain

$$\frac{\Gamma'(1/4)}{\Gamma(1/4)} = -\gamma - 3\log 2 - \frac{\pi}{2}.$$

The result for $\frac{\Gamma'(5/4)}{\Gamma(5/4)}$ now follows by substituting $z=\frac{1}{4}$ into the recurrence formula

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z}. \quad \square$$

This completes the solution of the problem.

Reference

1. E. C. Titchmarch. *The Theory of the Riemann Zeta-Function*. 2nd ed. Oxford: Clarendon Press, 1986.

Also solved by.-J. Seiffert and the proposer.

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