PRONIC FIBONACCI NUMBERS

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1. INTRODUCTION

"Pronic" is an old-fashioned term meaning "the product of two consecutive integers." (The reader will find the term indexed in [1], referring to some half-dozen articles.) In this paper we show that the only Fibonacci numbers that are the product of two consecutive integers are $F_0 = 0$ and $F_{\pm 3} = 2$.

The referee of this paper has called the author's attention to the prior publication (December 1996) of this result in Chinese (see Ming Luo [3]). However, because of the relative inaccessibility of the earlier result, the referee recommended publication of this article in the *Quarterly*.

If $F_n = r(r+1)$, then $4F_n + 1$ is a square. Our approach is to show that F_n, for $n \neq 0$, ± 3, is not a pronic number by finding an integer $w(n)$ such that $4F_n + 1$ is a quadratic nonresidue modulo $w(n)$. There is a sense in which this paper may be considered a companion paper to Ming Luo's article on triangular numbers in the sequence of Fibonacci numbers: If F_n is a pronic number, then F_n is two times a triangular number. We shall use two results from Luo's paper, and take advantage of the periodicity of the sequence modulo an appropriate integer $w(n)$, enabling us to prove our result through use of the Jacobi symbol $(4F_n + 1 | w(n))$ in a finite number of cases. Our main result is the following theorem.

Main Theorem: The Fibonacci number F_n is the product of two consecutive integers if and only if $n = -3$, 0, or 3.

2. IDENTITIES AND PRELIMINARY LEMMAS

Let n and m be integers and $\{L_n\}$ be the sequence of Lucas numbers. Properties (1) through (4) are well known, and (5) was established in Luo's paper [2].

$$F_{-n} = (-1)^{n+1} F_n. \tag{1}$$

$$L_{2n} = L_n^2 - 2(-1)^n. \tag{2}$$

$$F_{m+n} = F_m L_n - (-1)^n F_{m-n}. \tag{3}$$

$$2F_{m+n} = F_m L_n + F_n L_m. \tag{4}$$

If k is even, $3 \nmid k$, and $(a, L_k) = 1$, then

$$(\pm 4aF_{2k} + 1 | L_{2k}) = -(8aF_k \pm L_k | 64a^2 + 5). \tag{5}$$

If the period of $\{F_n\}$ modulo Q is t and $n \equiv m \pmod{t}$, then $F_n \equiv F_m \pmod{Q}$. We will use this fact in our proofs for the following pairs: $(t, Q) = (8, 3)$, $(20, 5)$, $(16, 7)$, $(24, 9)$, $(10, 11)$, $(40, 41)$, $(50, 101)$, $(50, 151)$, and $(100, 3001)$.

It should be noted that we have given the least period t modulo Q in each of the above pairs; however, $F_n \equiv F_m \pmod{Q}$ if $n \equiv m \pmod{ht}$ for any integer h.

Finally, we comment that it is well known that F_n and L_n are even if and only if $3|n$.

Lemma 1: For all integers k and m, and g odd,

$$F_{2kg+m} \equiv \begin{cases} F_{2k+m} \pmod{L_{2k}}, & \text{if } g \equiv 1 \pmod 4, \\ -F_{2k+m} \pmod{L_{2k}}, & \text{if } g \equiv 3 \pmod 4. \end{cases}$$

Proof: By (3),

$$F_{2kg+m} = F_{2k(g-1)+m}L_{2k} - (-1)^{2k}F_{2k(g-2)+m} \equiv -F_{2k(g-2)+m} \pmod{L_{2k}};$$

clearly,

$$F_{2kg+m} \equiv -F_{2k(g-2)+m} \equiv +F_{2k(g-4)+m} \equiv \cdots \equiv \pm F_{2k+m} \pmod{L_{2k}},$$

where the positive sign occurs if and only if $g \equiv 1 \pmod 4$.

Lemma 2: If $3 \nmid k$, then $F_{2k+3} \equiv 2F_{2k} \pmod{L_{2k}}$.

Proof: By (4),

$$2F_{2k+3} = F_{2k}L_3 + F_3L_{2k} \equiv F_{2k} \cdot 4 \pmod{L_{2k}},$$

implying the lemma, since L_{2k} is odd.

Lemma 3: If F_n is pronic, then $n \equiv 0$ or $\pm 3 \pmod 8$.

Proof: Assume $4F_n + 1$ is a square. Then $4F_n + 1$ is a quadratic residue modulo 3 and modulo 7. However, $4F_n + 1$ is a quadratic nonresidue modulo 3 if $n \equiv 1$, 2, or 7 (mod 8), and a nonresidue modulo 7 if $n \equiv 4$ or 12 (mod 16). If $n \equiv 6 \pmod 8$, then $n \equiv 6$, 14, or 22 (mod 24); but, for each of these n's, $4F_n + 1$ is a quadratic nonresidue modulo 9, establishing the lemma.

3. PROOFS OF THE THEOREMS

Theorem 1: If n is odd and $n \neq \pm 3$, then F_n is not pronic.

Proof: Assume n is odd, $n \neq \pm 3$, and F_n is pronic. By Lemma 3, $n \equiv \pm 3 \pmod 8$. First, we assume that $n \equiv 3 \pmod 8$. Then $n \equiv 3$, 11, 19, 27, or 35 (mod 40); however, $(4F_m + 1|Q) = -1$ for $(m, Q) = (11, 5)$, (19, 41), (27, 5), and (35, 11), implying $n \equiv 3 \pmod{40}$. Then $n \equiv 3$, 23, 43, 63, or 83 (mod 100). Proceeding as before, we find that $(4F_m + 1|Q) = -1$ for $(m, Q) = (23, 3001)$, (43, 101), (63, 151), and (83, 101). Hence, if $n \equiv 3 \pmod 8$, then $n \equiv 3 \pmod{100}$. Let $n = 2 \cdot 2^u \cdot 5^2 t + 3$, $u \geq 1$. Now, if $n = 2kg + 3$, $3 \nmid k$, and g is odd, then, by Lemmas 1 and 2,

$$(4F_n + 1|L_{2k}) = (\pm 8F_{2k} + 1|L_{2k}).$$

By (5), if k is even and $3 \nmid k$, then

$$(\pm 8F_{2k} + 1|L_{2k}) = -(16F_k \pm L_k|261) = -(16F_k \pm L_k|29).$$

In the proof of Luo's Lemma 2 (see [2]), it is shown that this Jacobi symbol is -1 for

$$\begin{array}{lll} k = 2^u & \text{and } g = 5^2 t & \text{if } u \equiv 0 \pmod 3, \\ k = 2^u \cdot 5^2 & \text{and } g = t & \text{if } u \equiv 1 \pmod 3, \\ k = 2^u \cdot 5 & \text{and } g = 5t & \text{if } u \equiv 2 \pmod 3. \end{array}$$

Thus, F_n is not pronic if $n \equiv 3 \pmod 8$.

Assume now that $n \equiv -3 \pmod 8$. By (1), $F_n = F_{-n}$ and, since $-n \equiv 3 \pmod 8$,

$$(4F_{-n}+1|L_{2k}) = -1$$

by the above proof. □

Lemma 4: If $u \geq 4$, then

(a) $F_{2^u} \equiv (-1)^u \cdot 21 \pmod{69}$ and $L_{2^u} \equiv -1 \pmod{69}$,

(b) $F_{2^u \cdot 5} \equiv (-1)^{u+1} \cdot 21 \pmod{69}$ and $L_{2^u \cdot 5} \equiv -1 \pmod{69}$.

Proof: $L_2 = 3$, $L_4 = 7$, $L_8 = 47$, $L_{16} = 2207 \equiv -1 \pmod{69}$ and, using (2), it follows by induction that $L_{2^u} \equiv -1 \pmod{69}$ for $u \geq 4$, Hence,

$$F_{2^u} = F_2 L_2 L_4 L_8 \ldots L_{2^{u-1}} \equiv 1 \cdot 3 \cdot 7 \cdot 47 \cdot (-1)^u \equiv (-1)^u \cdot 21 \pmod{69}.$$

Similarly, $L_{10}, L_{20}, L_{40}, L_{80} \equiv 54, 16, 47, -1 \pmod{69}$, respectively, and (b) readily follows. □

Proof of the Main Theorem: If $n = 0$ or ± 3, F_n is clearly the product of consecutive integers. Assume that $n \neq 0, \pm 3$, and F_n is pronic. By Lemma 3 and Theorem 1, $n \equiv 0 \pmod 8$; so $n \equiv 0, 8, 16, 24$, or $32 \pmod{40}$. But $(4F_m+1|Q) = -1$ for $(m, Q) = (8, 11)$, $(16, 41)$, $(24, 5)$, or $(32, 5)$, so $n \equiv 0 \pmod{40}$. Let $n = 2 \cdot 2^u \cdot 5t$, $u \geq 2$. By Lemma 1 and (5), if $n = 2kg$, $3 \nmid k$, k is even, and g is odd, then

$$(4F_n+1|L_{2k}) = (4F_{2kg}+1|L_{2k}) \equiv \begin{cases} -(8F_k+L_k|69), & \text{if } g \equiv 1 \pmod 4, \\ -(8F_k-L_k|69), & \text{if } g \equiv 3 \pmod 4. \end{cases}$$

Case 1: $t \equiv 1 \pmod 4$. Let

$$\begin{aligned} &k = 2^u \quad &&\text{and} \quad g = 5t \equiv 1 \pmod 4, \quad &&\text{if } u \text{ is odd, } u \neq 3 \text{ or } u = 2, \\ &k = 2^u \cdot 5 \quad &&\text{and} \quad g = t \equiv 1 \pmod 4, \quad &&\text{if } u \text{ is even, } u \neq 2 \text{ or } u = 3. \end{aligned}$$

If $u = 2$, $-(8F_k+L_k|69) = -(31|69) = -1$; if $u = 3$, $-(8F_k+L_k|69) = -(17|69) = -1$; if $u \geq 4$ and u is odd $(k = 2^u)$ or if u is even $(k = 2^u \cdot 5)$, then, by Lemma 4,

$$8F_k + L_k \equiv 8(-21) + -1 \equiv -169 \pmod{69}.$$

Hence, $-(8F_k+L_k|69) = -(-169|69) = -1$.

Case 2: $t \equiv 3 \pmod 4$. Let

$$\begin{aligned} &k = 2^u \quad &&\text{and} \quad g = 5t \equiv 3 \pmod 4, \quad &&\text{if } u \text{ is even or } u = 3, \\ &k = 2^u \cdot 5 \quad &&\text{and} \quad g = t \equiv 3 \pmod 4, \quad &&\text{if } u \text{ is odd, } \quad u \neq 3. \end{aligned}$$

If $u = 2$, $-(8F_k-L_k|69) = -(17|69) = -1$; if $u = 3$, $-(8F_k-L_k|69) = -(121|69) = -1$; if $u \geq 4$ and u is odd $(k = 2^u \cdot 5)$ or u is even $(k = 2^u)$, then, by Lemma 4,

$$8F_k - L_k \equiv 8 \cdot 21 - (-1) \equiv 169 \pmod{69}.$$

Hence, $-(8F_k-L_k|69) = -(169|69) = -1$. □

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