

# PRONIC FIBONACCI NUMBERS

Wayne L. McDaniel

University of Missouri-St. Louis, St. Louis, MO 63121

(Submitted April 1996-Final Revision June 1996)

## 1. INTRODUCTION

"Pronic" is an old-fashioned term meaning "the product of two consecutive integers." (The reader will find the term indexed in [1], referring to some half-dozen articles.) In this paper we show that the only Fibonacci numbers that are the product of two consecutive integers are  $F_0 = 0$  and  $F_{\pm 3} = 2$ .

The referee of this paper has called the author's attention to the prior publication (December 1996) of this result in Chinese (see Ming Luo [3]). However, because of the relative inaccessibility of the earlier result, the referee recommended publication of this article in the *Quarterly*.

If  $F_n = r(r+1)$ , then  $4F_n + 1$  is a square. Our approach is to show that  $F_n$ , for  $n \neq 0, \pm 3$ , is not a pronic number by finding an integer  $w(n)$  such that  $4F_n + 1$  is a quadratic nonresidue modulo  $w(n)$ . There is a sense in which this paper may be considered a companion paper to Ming Luo's article on triangular numbers in the sequence of Fibonacci numbers: If  $F_n$  is a pronic number, then  $F_n$  is two times a triangular number. We shall use two results from Luo's paper, and take advantage of the periodicity of the sequence modulo an appropriate integer  $w(n)$ , enabling us to prove our result through use of the Jacobi symbol  $(4F_n + 1 | w(n))$  in a finite number of cases. Our main result is the following theorem.

**Main Theorem:** The Fibonacci number  $F_n$  is the product of two consecutive integers if and only if  $n = -3, 0$ , or  $3$ .

## 2. IDENTITIES AND PRELIMINARY LEMMAS

Let  $n$  and  $m$  be integers and  $\{L_n\}$  be the sequence of Lucas numbers. Properties (1) through (4) are well known, and (5) was established in Luo's paper [2].

$$F_{-n} = (-1)^{n+1} F_n. \quad (1)$$

$$L_{2n} = L_n^2 - 2(-1)^n. \quad (2)$$

$$F_{m+n} = F_m L_n - (-1)^n F_{m-n}. \quad (3)$$

$$2F_{m+n} = F_m L_n + F_n L_m. \quad (4)$$

If  $k$  is even,  $3 \nmid k$ , and  $(a, L_k) = 1$ , then

$$(\pm 4aF_{2k} + 1 | L_{2k}) = -(8aF_k \pm L_k | 64a^2 + 5). \quad (5)$$

If the period of  $\{F_n\}$  modulo  $Q$  is  $t$  and  $n \equiv m \pmod{t}$ , then  $F_n \equiv F_m \pmod{Q}$ . We will use this fact in our proofs for the following pairs:  $(t, Q) = (8, 3)$ ,  $(20, 5)$ ,  $(16, 7)$ ,  $(24, 9)$ ,  $(10, 11)$ ,  $(40, 41)$ ,  $(50, 101)$ ,  $(50, 151)$ , and  $(100, 3001)$ .

It should be noted that we have given the least period  $t$  modulo  $Q$  in each of the above pairs; however,  $F_n \equiv F_m \pmod{Q}$  if  $n \equiv m \pmod{ht}$  for any integer  $h$ .

Finally, we comment that it is well known that  $F_n$  and  $L_n$  are even if and only if  $3|n$ .

**Lemma 1:** For all integers  $k$  and  $m$ , and  $g$  odd,

$$F_{2kg+m} \equiv \begin{cases} F_{2k+m} \pmod{L_{2k}}, & \text{if } g \equiv 1 \pmod{4}, \\ -F_{2k+m} \pmod{L_{2k}}, & \text{if } g \equiv 3 \pmod{4}. \end{cases}$$

**Proof:** By (3),

$$F_{2kg+m} = F_{2k(g-1)+m}L_{2k} - (-1)^{2k}F_{2k(g-2)+m} \equiv -F_{2k(g-2)+m} \pmod{L_{2k}};$$

clearly,

$$F_{2kg+m} \equiv -F_{2k(g-2)+m} \equiv +F_{2k(g-4)+m} \equiv \dots \equiv \pm F_{2k+m} \pmod{L_{2k}},$$

where the positive sign occurs if and only if  $g \equiv 1 \pmod{4}$ .

**Lemma 2:** If  $3 \nmid k$ , then  $F_{2k+3} \equiv 2F_{2k} \pmod{L_{2k}}$ .

**Proof:** By (4),

$$2F_{2k+3} = F_{2k}L_3 + F_3L_{2k} \equiv F_{2k} \cdot 4 \pmod{L_{2k}},$$

implying the lemma, since  $L_{2k}$  is odd.

**Lemma 3:** If  $F_n$  is pronic, then  $n \equiv 0$  or  $\pm 3 \pmod{8}$ .

**Proof:** Assume  $4F_n + 1$  is a square. Then  $4F_n + 1$  is a quadratic residue modulo 3 and modulo 7. However,  $4F_n + 1$  is a quadratic nonresidue modulo 3 if  $n \equiv 1, 2$ , or  $7 \pmod{8}$ , and a nonresidue modulo 7 if  $n \equiv 4$  or  $12 \pmod{16}$ . If  $n \equiv 6 \pmod{8}$ , then  $n \equiv 6, 14$ , or  $22 \pmod{24}$ ; but, for each of these  $n$ 's,  $4F_n + 1$  is a quadratic nonresidue modulo 9, establishing the lemma.

### 3. PROOFS OF THE THEOREMS

**Theorem 1:** If  $n$  is odd and  $n \neq \pm 3$ , then  $F_n$  is not pronic.

**Proof:** Assume  $n$  is odd,  $n \neq \pm 3$ , and  $F_n$  is pronic. By Lemma 3,  $n \equiv \pm 3 \pmod{8}$ . First, we assume that  $n \equiv 3 \pmod{8}$ . Then  $n \equiv 3, 11, 19, 27$ , or  $35 \pmod{40}$ ; however,  $(4F_m + 1|Q) = -1$  for  $(m, Q) = (11, 5), (19, 41), (27, 5)$ , and  $(35, 11)$ , implying  $n \equiv 3 \pmod{40}$ . Then  $n \equiv 3, 23, 43, 63$ , or  $83 \pmod{100}$ . Proceeding as before, we find that  $(4F_m + 1|Q) = -1$  for  $(m, Q) = (23, 3001), (43, 101), (63, 151)$ , and  $(83, 101)$ . Hence, if  $n \equiv 3 \pmod{8}$ , then  $n \equiv 3 \pmod{100}$ . Let  $n = 2 \cdot 2^u \cdot 5^2 t + 3$ ,  $u \geq 1$ . Now, if  $n = 2kg + 3$ ,  $3 \nmid k$ , and  $g$  is odd, then, by Lemmas 1 and 2,

$$(4F_n + 1|L_{2k}) = (\pm 8F_{2k} + 1|L_{2k}).$$

By (5), if  $k$  is even and  $3 \nmid k$ , then

$$(\pm 8F_{2k} + 1|L_{2k}) = -(16F_k \pm L_k | 261) = -(16F_k \pm L_k | 29).$$

In the proof of Luo's Lemma 2 (see [2]), it is shown that this Jacobi symbol is  $-1$  for

$$\begin{aligned} k = 2^u & \quad \text{and } g = 5^2 t \quad \text{if } u \equiv 0 \pmod{3}, \\ k = 2^u \cdot 5^2 & \quad \text{and } g = t \quad \text{if } u \equiv 1 \pmod{3}, \\ k = 2^u \cdot 5 & \quad \text{and } g = 5t \quad \text{if } u \equiv 2 \pmod{3}. \end{aligned}$$

Thus,  $F_n$  is not pronic if  $n \equiv 3 \pmod{8}$ .

Assume now that  $n \equiv -3 \pmod{8}$ . By (1),  $F_n = F_{-n}$  and, since  $-n \equiv 3 \pmod{8}$ ,

$$(4F_{-n} + 1 | L_{2k}) = -1$$

by the above proof.  $\square$

**Lemma 4:** If  $u \geq 4$ , then

(a)  $F_{2^u} \equiv (-1)^u \cdot 21 \pmod{69}$  and  $L_{2^u} \equiv -1 \pmod{69}$ ,

(b)  $F_{2^u \cdot 5} \equiv (-1)^{u+1} \cdot 21 \pmod{69}$  and  $L_{2^u \cdot 5} \equiv -1 \pmod{69}$ .

**Proof:**  $L_2 = 3$ ,  $L_4 = 7$ ,  $L_8 = 47$ ,  $L_{16} = 2207 \equiv -1 \pmod{69}$  and, using (2), it follows by induction that  $L_{2^u} \equiv -1 \pmod{69}$  for  $u \geq 4$ . Hence,

$$F_{2^u} = F_2 L_2 L_4 L_8 \dots L_{2^{u-1}} \equiv 1 \cdot 3 \cdot 7 \cdot 47 \cdot (-1)^u \equiv (-1)^u \cdot 21 \pmod{69}.$$

Similarly,  $L_{10}, L_{20}, L_{40}, L_{80} \equiv 54, 16, 47, -1 \pmod{69}$ , respectively, and (b) readily follows.  $\square$

**Proof of the Main Theorem:** If  $n = 0$  or  $\pm 3$ ,  $F_n$  is clearly the product of consecutive integers. Assume that  $n \neq 0, \pm 3$ , and  $F_n$  is pronic. By Lemma 3 and Theorem 1,  $n \equiv 0 \pmod{8}$ ; so  $n \equiv 0, 8, 16, 24$ , or  $32 \pmod{40}$ . But  $(4F_m + 1 | Q) = -1$  for  $(m, Q) = (8, 11), (16, 41), (24, 5)$ , or  $(32, 5)$ , so  $n \equiv 0 \pmod{40}$ . Let  $n = 2 \cdot 2^u \cdot 5t$ ,  $u \geq 2$ . By Lemma 1 and (5), if  $n = 2kg$ ,  $3 \nmid k$ ,  $k$  is even, and  $g$  is odd, then

$$(4F_n + 1 | L_{2k}) = (4F_{2kg} + 1 | L_{2k}) \equiv \begin{cases} -(8F_k + L_k | 69), & \text{if } g \equiv 1 \pmod{4}, \\ -(8F_k - L_k | 69), & \text{if } g \equiv 3 \pmod{4}. \end{cases}$$

**Case 1:**  $t \equiv 1 \pmod{4}$ . Let

$$k = 2^u \quad \text{and} \quad g = 5t \equiv 1 \pmod{4}, \quad \text{if } u \text{ is odd, } u \neq 3 \text{ or } u = 2,$$

$$k = 2^u \cdot 5 \quad \text{and} \quad g = t \equiv 1 \pmod{4}, \quad \text{if } u \text{ is even, } u \neq 2 \text{ or } u = 3.$$

If  $u = 2$ ,  $-(8F_k + L_k | 69) = -(31 | 69) = -1$ ; if  $u = 3$ ,  $-(8F_k + L_k | 69) = -(17 | 69) = -1$ ; if  $u \geq 4$  and  $u$  is odd ( $k = 2^u$ ) or if  $u$  is even ( $k = 2^u \cdot 5$ ), then, by Lemma 4,

$$8F_k + L_k \equiv 8(-21) + -1 \equiv -169 \pmod{69}.$$

Hence,  $-(8F_k + L_k | 69) = -(-169 | 69) = -1$ .

**Case 2:**  $t \equiv 3 \pmod{4}$ . Let

$$k = 2^u \quad \text{and} \quad g = 5t \equiv 3 \pmod{4}, \quad \text{if } u \text{ is even or } u = 3,$$

$$k = 2^u \cdot 5 \quad \text{and} \quad g = t \equiv 3 \pmod{4}, \quad \text{if } u \text{ is odd, } u \neq 3.$$

If  $u = 2$ ,  $-(8F_k - L_k | 69) = -(17 | 69) = -1$ ; if  $u = 3$ ,  $-(8F_k - L_k | 69) = -(121 | 69) = -1$ ; if  $u \geq 4$  and  $u$  is odd ( $k = 2^u \cdot 5$ ) or  $u$  is even ( $k = 2^u$ ), then, by Lemma 4,

$$8F_k - L_k \equiv 8 \cdot 21 - (-1) \equiv 169 \pmod{69}.$$

Hence,  $-(8F_k - L_k | 69) = -(169 | 69) = -1$ .  $\square$

**ACKNOWLEDGMENT**

The author wishes to express his thanks to the anonymous referee, and to the editor of this *Quarterly* who very graciously accepted the referee's recommendation to publish this paper in order to make the previously published results in Chinese more widely available. The author understands that the proofs in this paper and those in the earlier paper [3] (which he has not yet seen) are along similar lines but differ in detail.

**REFERENCES**

1. L. E. Dickson. *History of the Theory of Numbers*. New York: Chelsea, 1952.
2. Ming Luo. "On Triangular Fibonacci Numbers." *The Fibonacci Quarterly* 27.2 (1989):98-108.
3. Ming Luo. "Nearly Square Numbers in the Fibonacci and Lucas Sequences." *Journal of Chongqing Teachers College* 12.4 (1995):1-5. (In Chinese.)

AMS Classification Number: 11B39



**APPLICATIONS OF FIBONACCI NUMBERS**

**VOLUME 6**

*New Publication*

**Proceedings of The Sixth International Research Conference  
on Fibonacci Numbers and Their Applications,  
Washington State University, Pullman, Washington, USA, July 18-22, 1994**

*Edited by G. E. Bergum, A. N. Philippou, and A. F. Horadam*

This volume contains a selection of papers presented at the Sixth International Research Conference on Fibonacci Numbers and Their Applications. The topics covered include number patterns, linear recurrences, and the application of the Fibonacci Numbers to probability, statistics, differential equations, cryptography, computer science, and elementary number theory. Many of the papers included contain suggestions for other avenues of research.

For those interested in applications of number theory, statistics and probability, and numerical analysis in science and engineering:

**1996, 560 pp. ISBN 0-7923-3956-8  
Hardbound Dfl. 345.00 / £155.00 / US\$240.00**

AMS members are eligible for a 25% discount on this volume providing they order directly from the publisher. However, the bill must be prepaid by credit card, registered money order, or check. A letter must also be enclosed saying: "I am a member of the American Mathematical Society and am ordering the book for personal use."

**KLUWER ACADEMIC PUBLISHERS**

**P.O. Box 322, 3300 AH Dordrecht  
The Netherlands**

**P.O. Box 358, Accord Station  
Hingham, MA 02018-0358, U.S.A.**

Volumes 1-5 can also be purchased by writing to the same addresses.