DIVISIBILITY TESTS IN ℕ

James E. Voss
129 Woodland Avenue #7, San Rafael, CA 94901
(Submitted April 1996-Final Revision September 1996)

This article will develop a method to test divisibility of arbitrary natural numbers by certain fixed natural numbers. The well-known tests for divisibility by 3, 9, and 11 will be obtained as special cases of the theorem. Note that all the variables in the following theorem are integers.

Theorem: If $(s, 10) = 1$, $t \equiv 10^{-1} \pmod{s}$, $n = \sum_{k=0}^{r} 10^k a_k$, and $m = \sum_{k=0}^{r} t^{r-k} a_k$, then $s|n \Leftrightarrow s|m$.

Proof: We will expand n and use standard congruence properties:

$$\begin{aligned} n &= 10^r a_r + 10^{r-1} a_{r-1} + \cdots + 10 a_1 + a_0, \\ n &\equiv 10^r a_r + 10^{r-1} a_{r-1} + \cdots + 10 a_1 + a_0 \pmod{s}, \\ 10^{-r} n &\equiv a_r + 10^{-1} a_{r-1} + \cdots + 10^{1-r} a_1 + 10^{-r} a_0 \pmod{s}, \\ (10^{-1})^r n &\equiv a_r + 10^{-1} a_{r-1} + \cdots + (10^{-1})^{r-1} a_1 + (10^{-1})^r a_0 \pmod{s}, \\ t^r n &\equiv m \pmod{s}. \end{aligned}$$

Now $t \equiv 10^{-1} \pmod{s} \Rightarrow 10t \equiv 1 \pmod{s} \Rightarrow s|(10t-1) \Rightarrow zs = 10t - 1$ for some $z \in \mathbb{Z}$. Hence, $10t - zs = 1$, which implies $(s, t) = 1$.

The statement $t^r n \equiv m \pmod{s}$ allows us to conclude that $s|n \Rightarrow s|m$; with the additional fact that $(s, t) = 1$, we can conclude that $s|m \Rightarrow s|n$.

Remark: This theorem generates a divisibility test for any natural number s that is relatively prime to 10. The practicality of the test comes into play for s with an associated t value close to 0.

Divisibility Tests for Specific Natural Numbers

1. Let $s = 3$. Then $t \equiv 10^{-1}$ (mod 3) allows us to choose $t = 1$. Hence, $3|n \Leftrightarrow 3|m$, where $m = \sum_{k=0}^{r} a_k$
2. Let $s = 9$. Then $t \equiv 10^{-1}$ (mod 9) allows us to choose $t = 1$. Hence, $9|n \Leftrightarrow 9|m$, where $m = \sum_{k=0}^{r} a_k$.
3. Let $s = 11$. Then $t \equiv 10^{-1}$ (mod 11) allows us to choose $t = -1$. Hence, $11|n \Leftrightarrow 11|m$, where $m = \sum_{k=0}^{r} (-1)^{r-k} a_k$.
4. Let $s = 19$. Then $t \equiv 10^{-1}$ (mod 19) allows us to choose $t = 2$. Hence, $19|n \Leftrightarrow 19|m$, where $m = \sum_{k=0}^{r} 2^{r-k} a_k$.
5. Let $s = 7$. Then $t \equiv 10^{-1}$ (mod 7) allows us to choose $t = -2$. Hence, $7|n \Leftrightarrow 7|m$, where $m = \sum_{k=0}^{r} (-2)^{r-k} a_k$.
6. Let $s = 29$. Then $t \equiv 10^{-1}$ (mod 29) allows us to choose $t = 3$. Hence, $29|n \Leftrightarrow 29|m$, where $m = \sum_{k=0}^{r} 3^{r-k} a_k$.
7. Let $s = 31$. Then $t \equiv 10^{-1}$ (mod 31) allows us to choose $t = -3$. Hence, $31|n \Leftrightarrow 31|m$, where $m = \sum_{k=0}^{r} (-3)^{r-k} a_k$.

Specific Examples

Ex. 1: $n = 5232$ is divisible by $s = 3$ because we can take $t = 1$ and
$m = 5(1)^0 + 2(1)^1 + 3(1)^2 + 2(1)^3 = 5 + 2 + 3 + 2 = 12$ is divisible by 3.

Ex. 2: $n = 7119$ is divisible by $s = 9$ because we can take $t = 1$ and
$m = 7(1)^0 + 1(1)^1 + 1(1)^2 + 9(1)^3 = 7 + 1 + 1 + 9 = 18$ is divisible by 9.

Ex. 3: $n = 80916$ is divisible by $s = 11$ because we can take $t = -1$ and
$m = 8(-1)^0 + 0(-1)^1 + 9(-1)^2 + 1(-1)^3 + 6(-1)^4 = 8 - 0 + 9 - 1 + 6 = 22$ is divisible by 11.

Ex. 4: $n = 2242$ is divisible by $s = 19$ because we can take $t = 2$ and
$m = 2(2)^0 + 2(2)^1 + 4(2)^2 + 2(2)^3 = 2 + 4 + 16 + 16 = 38$ is divisible by 19.

Ex. 5: $n = 686$ is divisible by $s = 7$ because we can take $t = -2$ and
$m = 6(-2)^0 + 8(-2)^1 + 6(-2)^2 = 6 - 16 + 24 = 14$ is divisible by 7.

Ex. 6: $n = 4350$ is divisible by $s = 29$ because we can take $t = 3$ and
$m = 4(3)^0 + 3(3)^1 + 5(3)^2 + 0(3)^3 = 4 + 9 + 45 + 0 = 58$ is divisible by 29.

Ex. 7: $n = 527000$ is divisible by $s = 31$ because we can take $t = -3$ and
$m = 5(-3)^0 + 2(-3)^1 + 7(-3)^2 + 0(-3)^3 + 0(-3)^4 + 0(-3)^5 = 5 - 6 + 63 = 62$ is divisible by 31.

ACKNOWLEDGMENT

I would like to thank Dr. Neville Robbins for his advice and encouragement.

REFERENCE

1. N. Robbins. *Beginning Number Theory*. New York: Wm. C. Brown, 1993.

AMS Classification Number: 11A07

❖❖❖