BRAHMAGUPTA'S THEOREMS AND RECURRENCE RELATIONS

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1. INTRODUCTION

In this paper, we relate the positive integer solutions of the Diophantine equation of the type $x^2 - Dy^2 = \lambda$ with the generalized sequence of numbers $W_n(a,b;p,q)$ defined by Horadam [3]. We do this by utilizing the *principle of composition*, or *Bhavana*, first enunciated in the sixth century by the Indian astronomer and mathematician Brahmagupta while dealing with the integer solutions of the indeterminate equation $x^2 - Dy^2 = \lambda$, D being a positive integer that is not a perfect square and λ a positive or negative integer [1], [2]. Further, we show that all the integer solutions of the equation $x^2 - Dy^2 = \pm 1$ are related to the Chebyshev polynomials of the first and second kinds if the positive sign is taken in the equation and to the Pell and Pell-Lucas polynomials if the negative sign is taken in the equation. It may be of interest to note that Bhaskara II, another Indian mathematician of the twelfth century dealt extensively with the positive integer solutions of the equation $x^2 - Dy^2 = \lambda$, and gave an elegant method for finding a positive integer solution for an equation of the type $x^2 - Dy^2 = 1$. His technique is known as the *chakravala*, or cyclic method. Of course, this solution, in conjunction with Brahmagupta's method of composition, may be used to generate an infinite number of solutions to the equation $x^2 - Dy^2 = 1$ (see [1] and [2]). Among the many examples that Bhaskara II considered and solved are the equations $x^2 - 61y^2 = 1$ and $x^2 - 67y^2 = 1$. It is interesting to note that the equation $x^2 - 61y^2 = 1$ was proposed by Fermat to Frenicle in 1657 and that it was solved by Euler in 1732.

2. BRAHMAGUPTA'S THEOREMS

We first enunciate the following two theorems originally proposed by Brahmagupta.

Theorem 1 (Bhavana, or the Principle of Composition): If (x_1, y_1) is a solution of the equation $x^2 - Dy^2 = \lambda_1$ and (x_2, y_2) is a solution of the equation $x^2 - Dy^2 = \lambda_2$, then $(x_1x_2 \pm Dy_1y_2, x_1y_2 \pm x_2y_1)$ is a solution of the equation $x^2 - Dy^2 = \lambda_1\lambda_2$.

Theorem 2: If (x_1, y_1) is a solution of the equation $x^2 - Dy^2 = \pm d\lambda^2$ such that $\lambda | x_1$ and $\lambda | y_1$, then $(x_1 / \lambda,\ y_1 / \lambda)$ is a solution of $x^2 - Dy^2 = \pm d$.

As a consequence of Theorem 1, we can see that if (α, β) is a solution of the equation $x^2 - Dy^2 = -1$ then $(\alpha^2 - D\beta^2, 2\alpha\beta)$ is a solution of $x^2 - Dy^2 = 1$. It is well known that the equation $x^2 - Dy^2 = 1$ is always solvable in integers, while $x^2 - Dy^2 = -1$ may have no integer solutions [4], [5]. Bhaskara has shown that $x^2 - Dy^2 = -1$ has no integer solutions unless D is expressible as the sum of two squares [2].

Let us consider the different positive integer solutions of the equation

$$x^2 - Dy^2 = \lambda. \tag{1}$$

Let (a, b) be the "smallest" solution of (1), which is also referred to as the "fundamental" solution. Then, by repeated application of Theorem 1 (using the positive sign only), we can readily see that (x_n, y_n) is a solution of the equation

$$x_n^2 - Dy_n^2 = \lambda^n, \tag{2}$$

where (x_n, y_n) satisfy the recurrence relations

$$\begin{aligned} x_n &= ax_{n-1} + Dby_{n-1}, \\ y_n &= bx_{n-1} + ay_{n-1}. \end{aligned} \tag{3}$$

From (3), we see that (x_n, y_n) satisfy the difference equations

$$\begin{aligned} x_n &= 2ax_{n-1} - \lambda x_{n-2}, \qquad x_0 = 1,\ x_1 = a, \\ y_n &= 2ay_{n-1} - \lambda y_{n-2}, \qquad y_0 = 0,\ y_1 = b. \end{aligned} \tag{4}$$

Hence, (x_n, y_n) may be expressed in terms of the generalized sequence $W_n(a, b; p, q)$ defined by Horadam [3] in the form

$$x_n = W_n(1, a; 2a, \lambda), \qquad y_n = W_n(0, b; 2a, \lambda), \tag{5}$$

where

$$W_n = pW_{n-1} - qW_{n-2} \ (n \ge 2), \qquad W_0 = a,\ W_1 = b. \tag{6}$$

The difference equations given by (4) have been established recently by Suryanarayan [6], who has very appropriately called x_n and y_n "Brahmagupta polynomials." In the same context, it is appropriate to call equation (1) "Bhaskara's equation" (rather than a Pellian equation), since Pell has made no contribution to this topic, while Bhaskara (in the twelfth century) was the first to present a method for finding a positive integer solution of (1) when $\lambda = 1$.

Using the properties of the sequence $W_n(a, b; p, q)$, it is easy to show that

$$x_n = \frac{1}{2}v_n(2a, \lambda), \qquad y_n = bu_n(2a, \lambda), \tag{7}$$

where $u_n(x, y)$ and $v_n(x, y)$ are generalized polynomials in two variables defined by

$$u_n(x, y) = xu_{n-1}(x, y) - yu_{n-2}(x, y), \quad u_0(x, y) = 0,\ u_1(x, y) = 1, \tag{8}$$

and

$$v_n(x, y) = xv_{n-1}(x, y) - yv_{n-2}(x, y), \quad v_0(x, y) = 2,\ v_1(x, y) = x. \tag{9}$$

A number of properties of the polynomials $u_n(x, y)$ and $v_n(x, y)$ have been derived recently [7]. In particular, we have

$$\begin{aligned} u_n(x, y) &= \sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-r-1}{r} x^{n-2r-1} y^r, \\ v_n(x, y) &= \sum_{r=0}^{[n/2]} (-1)^r \frac{n}{n-r} \binom{n-r}{r} x^{n-2r} y^r. \end{aligned} \tag{10}$$

It has been shown by Suryanarayan [6] that the Brahmagupta polynomials x_n and y_n have the property that

$$Y(s) = sbe^{2X(s)}, \tag{11}$$

where

$$X(s) = \sum_1^\infty \frac{x_n}{n} s^n, \qquad Y(s) = \sum_1^\infty y_n s^n. \tag{12}$$

Since x_n and y_n are related to the polynomials u_n and v_n by (7), it follows that

$$U_n(s) = e^{V_n(s)}, \tag{13}$$

where

$$U_n(s) = \sum_1^\infty u_n s^{n-1}, \qquad V_n(s) = \sum_1^\infty v_n s^n. \tag{14}$$

Relations similar to (13) hold good between Fibonacci and Lucas polynomials $F_n(x)$ and $L_n(x)$, Pell and Pell-Lucas polynomials $P_n(x)$ and $Q_n(x)$, Morgan-Voyce polynomials $B_n(x)$ and $C_n(x)$, Chebyshev polynomials of the first and second kind $T_n(x)$ and $S_n(x)$, etc., since

$$\begin{aligned} F_n(x) &= u_n(x, -1), & L_n(x) &= v_n(x, -1); \\ P_n(x) &= u_n(2x, -1), & Q_n(x) &= v_n(2x, -1); \\ B_n(x) &= u_{n+1}(x+2, 1), & C_n(x) &= v_n(x+2, 1); \\ S_n(x) &= u_n(2x, 1), & 2T_n(x) &= v_n(2x, 1). \end{aligned} \tag{15}$$

3. BHASKARA'S EQUATION WITH $\lambda = 1$ $(x^2 - Dy^2 = 1)$

Letting $\lambda = 1$ in (7), we see that the positive integer solutions of the equation

$$x^2 - Dy^2 = 1 \tag{16}$$

are given by

$$x_n = \frac{1}{2} v_n(2a, 1), \qquad y_n = b u_n(2a, 1), \tag{17}$$

$(n = 1, 2, 3, \ldots)$, where (a, b) is the fundamental solution of equation (16). Since $u_n(2x, 1) = S_n(x)$ and $v_n(2x, 1) = 2T_n(x)$, the Chebyshev polynomials of the first and second kind, we see that the positive integer solutions of equation (16) are given by

$$x_n = T_n(a), \qquad y_n = bS_n(a). \tag{18}$$

4. BHASKARA'S EQUATION WITH $\lambda = -1$ $(x^2 - Dy^2 = -1)$

It is well known that it may not always be possible to obtain positive integer solutions to the equation

$$x^2 - Dy^2 = -1. \tag{19}$$

In fact, it is not solvable unless the length of the period in the continued fraction expansion of $\sqrt{D}$ is odd [1]. Let us assume so, and let the fundamental solution of (19) be (a, b). Then, from (2) and (4), we have that (x_n, y_n) is a solution of the equation

$$x^2 - Dy^2 = (-1)^n, \tag{20}$$

where

$$\begin{aligned} x_n &= 2ax_{n-1} + x_{n-2}, \quad x_0 = 1,\ x_1 = a, \\ y_n &= 2ay_{n-1} + y_{n-2}, \quad y_0 = 0,\ y_1 = b. \end{aligned} \tag{21}$$

Hence,

$$x_n = \frac{1}{2} v_n(2a, -1), \qquad y_n = bu_n(2a, -1). \tag{22}$$

Thus, (22) gives the various solutions for the equations $x^2 - Dy^2 = -1$ and $x^2 - Dy^2 = 1$, respectively, depending on whether n is odd or even. Since $u_n(2x, -1) = P_n(x)$ and $v_n(2x, -1) = Q_n(x)$, where $P_n(x)$ and $Q_n(x)$ are the Pell and Pell-Lucas polynomials, respectively, we may rewrite (22) as

$$x_n = \frac{1}{2} Q_n(a), \qquad y_n = bP_n(a). \tag{23}$$

Now we see that

$$x_n = \frac{1}{2} Q_{2n-1}(a), \qquad y_n = bP_{2n-1}(a) \tag{24}$$

are the various integer solutions of $x^2 - Dy^2 = -1$, while

$$x_n = \frac{1}{2} Q_{2n}(a), \qquad y_n = bP_{2n}(a) \tag{25}$$

are those of $x^2 - Dy^2 = 1$, where (a, b) is the fundamental solution of $x^2 - Dy^2 = -1$.

Hence, we see that all the integer solutions of $x^2 - Dy^2 = 1$ are expressible in terms of the Chebyshev polynomials of the first and second kinds, while those of $x^2 - Dy^2 = -1$ are expressible in terms of the Pell and Pell-Lucas polynomials.

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