BOUNDS ON THE FIBONACCI NUMBER OF A MAXIMAL OUTERPLANAR GRAPH

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1. INTRODUCTION

All graphs in this article are finite, undirected, without loops or multiple edges. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$. The *complement* in G of a subgraph H is the subgraph of G obtained by deleting all edges in H. The *join* $G_1 \vee G_2$ of two graphs G_1 and G_2 is obtained by adding an edge from each vertex in G_1 to each vertex in G_2. Let K_n be the complete graph and P_n the path on n vertices.

The concept *Fibonacci number f* of a simple graph G refers to the number of subsets S of $V(G)$ such that no two vertices in S are adjacent [5]. Accordingly, the total number of subsets of $\{1, 2, \ldots, n\}$ such that no two elements are adjacent is F_{n+1}, the $(n+1)^{\text{th}}$ Fibonacci number.

2. THE FIBONACCI NUMBER OF A GRAPH

The following propositions can be found in [1], [2], and [3].

(a) $f(P_n) = F_{n+1}$.

(b) Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs with $E_1 \subseteq E_2$, then $f(G_2) \le f(G_1)$.

(c) Let $G = (V, E)$ be a graph with $u_1, u_2, \ldots, u_s$ vertices not contained in V. If $G_1 = (V_1, E_1)$ denotes the graph with $V_1 = V \cup \{u_1, \ldots, u_s\}$ and $E_1 = E \cup \{\{u_1, v_j\}, 1 \le i \le s, v_j \in V\}$, then $f(G_1) = f(G) + 2^s - 1$.

(d) A fan on k vertices, denoted by N_k, is the graph obtained from path P_{k-1} by making vertex 1 adjacent to every vertex of P_{k-1}, we have $f(N_k) = F_k + 1$.

(e) If T is a tree on n vertices, then $F_{n+1} \le f(T) \le 2^{n-1} + 1$. The upper and lower bounds are assumed by the stars S_n and paths P_n, where $f(S_n) = 2^{n-1} + 1$ and $f(P_n) = F_{n+1}$.

(f) If G_1 and G_2 are disjoint graphs, then $f(G_1 \cup G_2) = f(G_1) \cdot f(G_2)$.

3. THE SPECTRUM OF A GRAPH

The spectral radius $r(G)$ is the largest eigenvalue of its adjacency matrix $A(G)$. For $n \ge 4$ let $\mathcal{H}_n$ be the class of all *maximal outerplanar graphs* (Mops for short) on n vertices. If $G \in \mathcal{H}_n$, then G has at least two vertices of degree 2, has a plane representation as an n-gon triangulated by $n-3$ chords, and the boundary of this n-gon is the unique Hamiltonian cycle Z of G. As in [4], we let P_n^2 denote the graph obtained from P_n by adding new edges joining all pairs of vertices at a distance 2 apart. An *internal triangle* is a triangle in a Mop with no edge on the outer face. Let $\mathcal{G}_n$ be the subclass of all Mops in $\mathcal{H}_n$ with no internal triangle. Rowlinson [6] proved that $K_1 \vee P_{n-1}$ is the unique graph in $\mathcal{G}_n$ with maximal spectral radius. He also proved the uniqueness of P_n^2 with minimal $r(G)$ for all graphs in $\mathcal{G}_n$. In [6], Cvetković and Rowlinson conjectured that $K_1 \vee P_{n-1}$ with spectral radius very close to $1+\sqrt{n}$ is the unique graph with the largest radius among all Mops in $\mathcal{H}_n$. In [2], Cao and Vince showed that the largest eigenvalue of $K_1 \vee P_{n-1}$ is between $1+\sqrt{n}-\frac{1}{2+n-2\sqrt{n}}$ and $1+\sqrt{n}$. This result comes close to confirming the conjecture of Rowlinson and Cvetković but does not settle it.

We will show that these two graphs $K_1 \vee P_{n-1}$ and P_n^2 are extremal and unique in $\mathcal{H}_n$ with respect to their Fibonacci numbers.

All Mops of order 8 are shown in Figure 1. Each Mop is labelled by its spectral radius r and Fibonacci number f.

4. THE UPPER BOUND

We established in [1] an upper bound on f of all Mops in $\mathcal{H}_n$ as in the following theorem.

Theorem 1: The Fibonacci number $f(G)$ of a maximal outerplanar graph G of order $n \geq 3$ is bounded above by F_n+1. Moreover, this upper bound is best possible.

The upper bound in Theorem 1 is realized by the Mop $K_1 \vee P_{n-1}$. Here, we prove that this Mop is unique.

Theorem 2: $K_1 \vee P_{n-1}$ is unique in $\mathcal{H}_n$.

Proof: We suppose that $n \geq 6$ because, if $n\{4,5\}$, then $K_1 \vee P_{n-1} = P_n^2$ and $\mathcal{H}_n$ contains only one graph. We continue the proof by induction on n. Assume uniqueness for all Mops of order less than n, and let G be a Mop of order n, $G \neq K_1 \vee P_{n-1}$. There exists a vertex v of degree 2 in G. We consider two families of subsets of $V(G)$. Each subset in the first family contains v, whereas v is not in any subset of the second family. Let u and w be neighbors of v in G. Deleting u and w, we obtain the outerplanar graph $G_{u,w}$ of order $n-3$ and the isolated vertex v. Since G is a triangulation of a polygon, $G_{u,w}$ contains a path P_{n-3} of length $n-4$.

Note that v can be chosen so that $d(u)+d(w)$ in G is minimum. Also, since $G \neq K_1 \vee P_{n-1}$, then $G_{u,w} \neq P_{n-3}$. Moreover, P_{n-3} is a proper subgraph of $G_{u,w}$. By Proposition (a),

$$f(P_{n-3}) = F_{n-2}$$

and, since v is a member of every subset of $V(G)$,

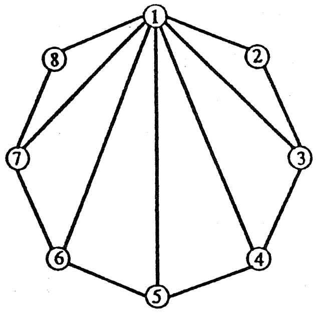
$$f(P_{n-3} \cup \{v\}) = f(P_{n-3}).$$

Now, by Proposition (b),

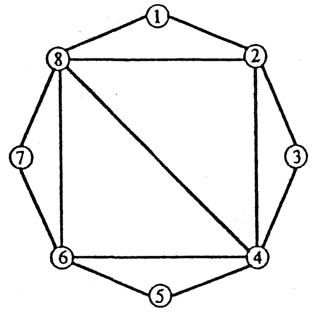
$$f(G_{u,w}) < F_{n-2}.$$

Next, we consider those admissible subsets of $V(G)$ not containing v. Let G_v be the remaining graph of order $n-1$ after deleting v. G_v is maximal outerplanar of order $n-1$. By the induction hypothesis, $K_1 \vee P_{n-2}$ is unique in $\mathcal{H}_{n-1}$, and this implies that $f(G_v)$ is strictly less than $F_{n-1}+1$. Combining the above results, we have

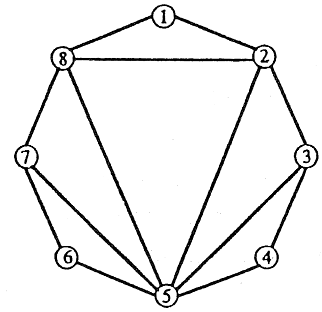
$$f(G) = f(G_{u,w}) + F(G_v) < F_{n-2} + F_{n-1} + 1 < F_n + 1.$$



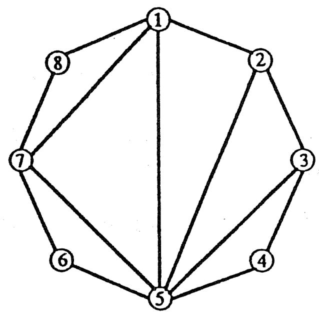
(1) $r \cong 3.676, f = 35$



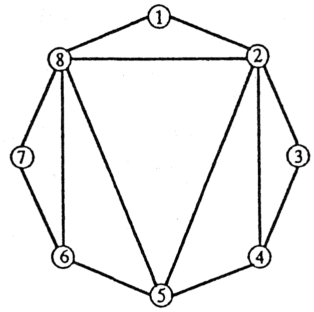
(2) $r \cong 3.646, f = 32$



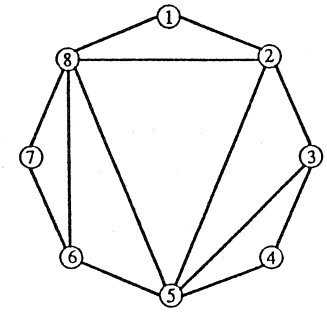
(3) $r \cong 3.639, f = 32$



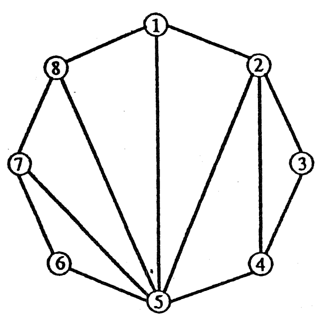
(4) $r \cong 3.63, f = 32$



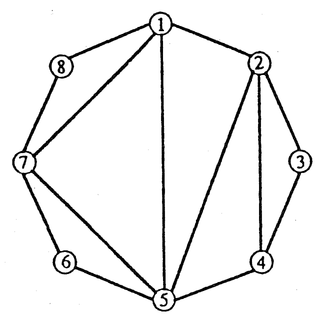
(5) $r \cong 3.602, f = 32$



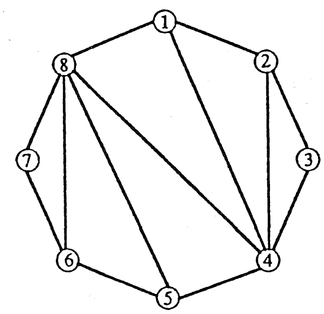
(6) $r \cong 3.608, f = 31$



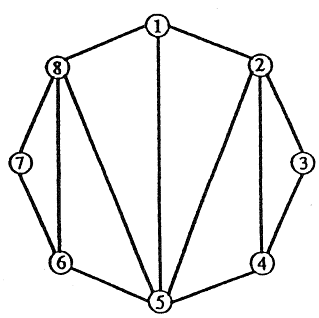
(7) $r \cong 3.590, f = 32$



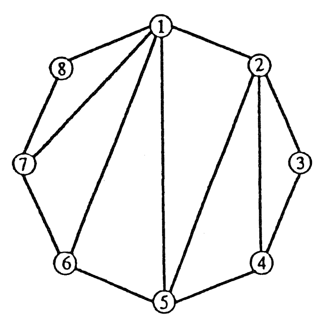
(8) $r \cong 3.570, f = 30$



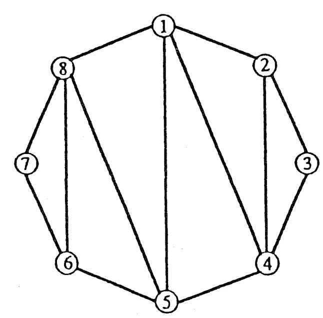
(9) $r \cong 3.550, f = 31$



(10) $r \cong 3.520, f = 29$



(11) $r \cong 3.516, f = 29$



(12) $r \cong 3.477, f = 28$

FIGURE 1. Twelve Mops with Their Spectral Radii and Fibonacci Numbers Indicated

5. THE LOWER BOUND

For the lower bound in $\mathcal{H}_n$, we let $H_n = P_n^2$, $n \geq 6$. These Mops H_n satisfy a recurrence relation $f(H_n) = f(H_{n-1}) + f(H_{n-3})$, whose solution h_n is

$$h_n = \left[\frac{u+v+10}{3u+3v}\right]\left[\frac{u+v+1}{3}\right]^n + \left[\frac{u+v-5}{3u+3v}\right]\left[-\frac{u+v-2}{6}+\frac{u-v}{6}\sqrt{3}i\right]^n + \left[\frac{u+v-5}{3u+3v}\right]\left[-\frac{u+v-2}{6}-\frac{u-v}{6}\sqrt{3}i\right]^n,$$

where

$$u = \sqrt[3]{\frac{29+3\sqrt{93}}{2}} \quad \text{and} \quad v = \sqrt[3]{\frac{29-3\sqrt{93}}{2}}.$$

After simplification, we have

$$h_n \cong (1.3134\ldots)(1.4655\ldots)^n.$$

Figure 2 shows a configuration of H_n for the even and odd cases.

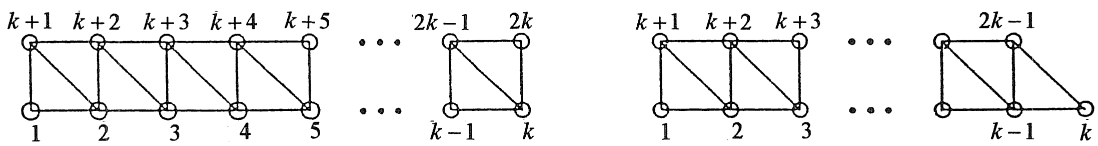


FIGURE 2. H_n Satisfies the Lower Bound

Theorem 3: The Fibonacci number $f(G)$ of a maximal outerplanar graph G of order $n \geq 3$ is bounded below by $f(P_n^2)$. Moreover, P_n^2 is unique.

Proof: As in the proof of Theorem 2, we suppose $n \geq 6$. We will prove the theorem by induction on n. The result is obvious for graphs of small order. Assume the validity of the theorem for all Mops of order less than n and let G be a Mop of order n where $G \neq P_n^2$. Each Mop has at least two vertices of degree 2. Suppose v is a vertex of degree 2 and u and w are adjacent to v. Since there are at least two choices of v, we will choose vertex v such that $d(u)+d(w)$ is maximum. We consider two families of subsets of $V(G)$. Each subset in the first family contains v, whereas v is not in any subset of the second family. Deleting u and w, we obtain the outerplanar subgraph $G_{u,w}$ of order $n-3$ and the isolated vertex v. Now $G_{u,w}$ is not maximal. We construct the Mop $G_{u,w}^*$ containing $G_{u,w}$ by adding edges in such a way that $\Delta(G_{u,w}^*) \geq 5$. This construction is always possible due to our choice of the vertex v. Thus, $G_{u,w}^* \neq P_{n-3}^2$ and, by the induction hypothesis,

$$f(G_{u,w}) > f(G_{u,w}^*) \geq f(P_{n-3}^2) = f(H_{n-3}). \qquad (*)$$

Next, we consider those sets of $V(G)$ not containing v. Let G_v be the remaining graph of order $n-1$ after deleting the vertex v. G_v is a Mop. By the induction hypothesis

$$f(G_v) \geq f(P_{n-1}^2) = f(H_{n-1}). \qquad (**)$$

Combining (*) and (**), we have

$$f(G) = f(G_v) + f(G_{u,w}) > f(H_{n-1}) + f(H_{n-3}) = f(H_n) = f(P_n^2).$$

We summarize our results for $n \le 20$ in Table 1.

TABLE 1. The Fibonacci Numbers F_n, $f(K_1 \vee P_{n-1})$ and $f(P_n^2)$ for $n \le 20$

n	F_n	$f(K_1 \vee P_{n-1})$	$f(P_n^2)$
0	1	1	1
1	1	2	2
2	2	3	3
3	3	4	4
4	5	6	6
5	8	9	9
6	13	14	13
7	21	22	19
8	34	35	28
9	55	56	41
10	89	90	60
11	144	145	88
12	233	234	129
13	377	378	189
14	610	611	277
15	987	988	406
16	1597	1598	595
17	2584	2585	872
18	4181	4182	1278
19	6765	6766	1873
20	10946	10947	2745

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